

On the General Theory of Elastic Stability

R. V. Southwell

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V. *On the General Theory of Elastic Stability.*By R. V. SOUTHWELL, *B.A.*, *Fellow of Trinity College, Cambridge.**Communicated by Prof. A. E. H. LOVE, F.R.S.*

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Introduction and Summary of Paper.

PROBLEMS which deal with the stability of bodies in equilibrium under stress are so distinct from the ordinary applications of the theory of elasticity that it is legitimate to regard them as forming a special branch of the subject. In every other case we

are concerned with the integration of certain differential equations, fundamentally the same for all problems, and the satisfaction of certain boundary conditions; and by a theorem due to KIRCHHOFF* we are entitled to assume that any solution which we may discover is unique. In these problems we are confronted with the possibility of two or more configurations of equilibrium, and we have to determine the conditions which must be satisfied in order that the equilibrium of any given configuration may be stable.

The development of both branches has proceeded upon similar lines. That is to say, the earliest discussions were concerned with the solution of isolated examples rather than with the formulation of general ideas. In the case of elastic stability, a comprehensive theory was not propounded until the problem of the straight strut had been investigated by EULER,† that of the circular ring under radial pressure by M. LÉVY‡ and G. H. HALPHEN,§ and A. G. GREENHILL had discussed the stability of a straight rod in equilibrium under its own weight,|| under twisting couples, and when rotating.¶

In a paper which has become the foundation of the theory in its existing form,** G. H. BRYAN has brought these isolated problems for the first time within the range of a single generalization. Examining the conditions under which KIRCHHOFF'S theorem of determinacy may fail, he was led to the conclusion that instability is only possible in the case of such bodies as thin rods, plates, or shells, and in these only when types of distortion can occur which do not involve extension of the central line or middle surface, so that it is legitimate to discuss any problem in elastic stability by methods which have been devised for the approximate treatment of such bodies. He showed, moreover, that the stability of the equilibrium of any given configuration depends upon the condition that the potential energy shall be a minimum in that configuration.

A closer examination of BRYAN'S theory suggests that some of the conclusions which have been drawn from it are scarcely warranted. The contention that no closed shell can fail by instability, because any distortion would involve extension of the middle surface, will be discussed later.†† For our present purpose it is sufficient to remark that the whole theory is based upon the assumption that the strains occurring previously to collapse must be kept to the extremely narrow limits within which, in the case of ordinary materials, HOOKE'S Law is satisfied. This assumption, of course, expresses a restriction necessarily imposed upon the range of practical

* A. E. H. LOVE, 'Mathematical Theory of Elasticity' (second edition), §118.

† 'Hist. Acad. Berlin,' XIII. (1757), p. 252.

‡ 'LIUVILLE'S Journal,' X. (1884), p. 5.

§ 'Comptes Rendus,' XCVIII. (1884), p. 422.

|| 'Proc. Camb. Phil. Soc.,' IV. (1881), p. 65.

¶ 'Proc. Inst. Mech. Eng.,' 1883, p. 182.

** 'Proc. Camb. Phil. Soc.,' VI. (1888), p. 199.

†† Cf. pp. 222, 236.

problems which can be treated by the ordinary theory of elasticity; but it is not legitimate to conclude that instability is only possible, even if its conditions were only calculable, in the case of materials which obey HOOKE'S Law, and there is no warrant for the employment of "crushing formulæ" in the design of short struts and thick boiler flues.*

A more serious weakness in the existing theory of elastic stability, when regarded from the mathematical standpoint, is the fact that the methods which it employs are admittedly only approximate. The higher the elastic limit† of the material under consideration, the less adequate are these methods to deal with the whole range of problems which should come within its scope. In fact, we are faced with the anomaly that, while in its ordinary applications the theory of elasticity is not concerned with the conception of an elastic limit, in questions of stability the existence of finite limits is an essential condition for the adequacy of its results. In an ideal material, possessing perfect elasticity combined with unlimited strength, types of instability could occur with which existing methods would be quite insufficient to deal.

The theory of elastic stability is thus in much the same position as that of the ordinary theory of elasticity before the discovery of the general equations, and one aim of the present paper is to remedy its defects by the investigation of general equations, which may be termed "Equations of Neutral Equilibrium," and which express the condition that a given configuration may be one of limiting equilibrium. These equations are universally applicable only to ideal material of indefinite strength, and the possibility of elastic break-down must receive separate investigation; but they are also applicable, even with materials of finite strength, to any problem which comes within the restrictions imposed by BRYAN'S discussion, and therefore enable us to test the accuracy of his treatment of problems, such as that of the boiler flue, for which the ordinary Theory of Thin Shells has been thought insufficiently rigorous.‡

In every problem of this paper it is found that the Theory of Thin Shells gives a solution which is correct as a first approximation, and the practical advantages of the new method of investigation are, therefore, not immediately apparent. But it must be remembered that the approximate theory of thin plates and shells has not as yet been rigorously established, and that much work has recently been undertaken with the object of testing it by comparison with accurate solutions of isolated problems.§ Now in finding conditions for the neutrality of the equilibrium of any given configuration we are at the same time obtaining the solution of a statical problem; for a configuration of slight distortion from the equilibrium position will also be one

* W. C. UNWIN, 'Elements of Machine Design' (1909), Part I., p. 111; S. E. SLOCUM, "The Collapse of Tubes under External Pressure," 'Engineering,' January 8, 1909.

† By "elastic limit" is intended, here and throughout this paper, the limit of linear elasticity.

‡ Cf. pp. 210, 224.

§ LOVE, *op. cit.*, Introduction, p. 29, and Chapter XXII.

of equilibrium. Hence every solution which we can obtain will add to the number of these "test cases," which has not hitherto included solutions for any but plane plates.

A far more important advantage of the new method, from the practical point of view, is the accuracy with which it follows the actual "stress history" in a body which fails by instability under a gradually increasing stress. In cases where instability precedes elastic break-down this difference of method is not important; but for the discussion of instability in overstrained material, where the stress-strain relations are intimately dependent upon the previous stress history, its introduction is absolutely necessary.

The extension of EULER'S theory to struts of practical dimensions and materials, which forms the conclusion of this paper, suggests a large and new field for investigation. The number of similar cases which can be treated, in the existing state of our knowledge of plastic strain, is very small, and indications are given below of the questions which still require an answer; there is reason to believe that the requisite experimental research would not present insuperable difficulties, and that we may hope in the future to obtain an adequate theory of experimental results which are at present very little understood.

EQUATIONS OF NEUTRAL EQUILIBRIUM IN RECTANGULAR CO-ORDINATES.

Method of Derivation.

The question of stability arises in regard to any system in which there is a possibility of slight displacement from the configuration of equilibrium. This possibility may be afforded either by a more or less limited degree of mechanical freedom—in which case the problem is one of statical stability, and practically unaffected by

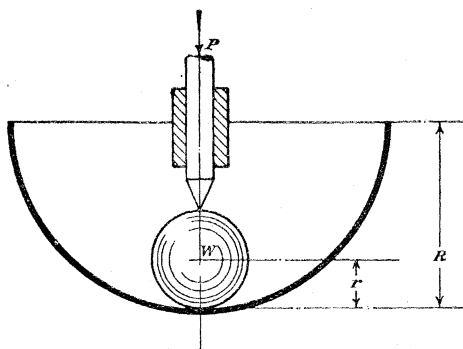


Fig. 1.

the tendency, which any actual body displays, to distort under the influence of applied forces; or it may be due, more or less entirely, to this tendency. In the latter case the problem is one of elastic stability, and must be treated by distinct methods. There is, however, no essential difference between the two types of

instability, and a general discussion of the elastic type may be very conveniently illustrated by reference to a mechanical example.

In this connection we may consider the system illustrated by fig. 1, in which a uniform heavy sphere rests in equilibrium within a hemispherical bowl, under the action of its own weight and of the pressure exerted by a pointed plunger, which is free to move in a vertical line through the centre of the bowl. This system has been chosen for the illustration which it affords of collapse under a definite "critical loading." In this it bears an unusual resemblance to examples of elastic instability—the stability of most mechanical systems being dependent solely upon the relative dimensions of their members. In the absence of friction, we find that the equilibrium will become unstable as the load on the plunger is increased through a critical value given by

$$P_c = \frac{Wr}{R-2r}, \quad \dots \dots \dots (1)$$

where

W is the weight of the sphere,

r is the radius of the sphere,

and

R is the radius of the bowl.

The above solution rests upon the assumption that the sphere, bowl and plunger are absolutely smooth and rigid, and the possibility of slight displacement is afforded by the freedom of the sphere to take up any position of contact with the bowl. To discuss the equilibrium of the sphere in the position illustrated we must consider the forces which act upon it in a position of slight displacement. These include two systems, one tending to restore the initial conditions, the other tending to increase the distortion, and stability depends upon the relative magnitude of the two effects. We may investigate the problem by three methods, fundamentally equivalent, which are described below:—

- (1) *The Energy Method.*—We may derive expressions for the potential energy of the system in a position of slight displacement from the equilibrium position. The condition of stability requires that the expression for the potential energy shall have a minimum value in the equilibrium position.
- (2) *The Method of Vibrations.*—We assume that the slight displacement has been effected by any cause, and investigate the types of vibration possible to the system when this cause is removed. The condition of stability requires that all such types shall have real periods.
- (3) *The Statical Method.*—We confine our attention to the special case in which the stability of the equilibrium position is *neutral*. In this case there must exist some type of displacement for which the collapsing and restoring effects, discussed above, are exactly balanced, so that it may be maintained by the original system of applied forces. We have, therefore, to find conditions for the equilibrium of a configuration of small displacement, under the given system of applied forces.

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Any of these methods is valid for the investigation of elastic stability, and all have in fact been employed, the displacement considered being that of the central-line or middle-surface of the rod or shell, and the resultant actions over cross-sections being derived in terms of this displacement, by the approximate theory first suggested by KIRCHHOFF. The third method is generally found to be preferable, and is the basis of the investigation to be described below, but the actual procedure will be found to possess one or two novel features.

In the first place, an endeavour will be made to dispense with the assumption that elastic break-down occurs at very small values of the strains; instead, we shall deal with an ideal material possessing perfect elasticity combined with unlimited strength. Such a material *could not fail, unless by instability*, and our problems will no longer be confined to thin rods, plates, or shells. It follows that we can only obtain sufficient accuracy in our conditions for neutral stability by deriving them with reference to a *volume-element* of the material.

Further, since instability will in some cases not occur until the strains in the material have reached finite values, we shall have to introduce an unusual precision into our ideas of stress and strain. The discussion of finite strain is merely a problem in kinematics, and has been worked out with some completeness*; but the corresponding stress-strain relations in our ideal material are necessarily less certain, since they must be based upon experiments in which only small strains are permissible.

For example, if we assume that HOOKE'S Law is satisfied at all stresses, we must decide whether our definition of stress is to be

$$\text{Lt. } \left[\frac{\text{Total action over an element of surface}}{\text{Original area of that surface}} \right]$$

or

$$\text{Lt. } \left[\frac{\text{Total action over the surface}}{\text{Area of that surface after distortion}} \right].$$

For the ordinary purposes of elastic theory the two definitions may be regarded as equivalent, and the distinction is too fine to be settled experimentally. In the absence of any generally-accepted molecular theory which might indicate the correct result, it seems legitimate to make the simplest possible assumptions which do not involve self-contradictions, and which yield the usual results when the strains are very small.

It may be shown† that in a distortion of any magnitude three orthogonal linear elements issue from any point after distortion, which were also orthogonal in the unstrained configuration, and that these linear elements undergo stationary (*maximum* or *minimum* or *minimax*) extension. Hence an elementary parallelepiped constructed at the point, with sides parallel to these linear elements, undergoes no change of angle in the distortion. It is clear that only normal stresses will act upon its faces

* For a discussion of the theory, with references, see LOVE, *op. cit.*, Appendix to Chapter I.

† LOVE, *op. cit.*, §§ 26, 27.

after distortion, and that if these stresses be expressed in terms of the extensions of the sides we have complete relations between stress and strain.

We shall therefore assume that *these principal stresses and principal strains, whatever their magnitude, are connected by the ordinary equations of HOOKE'S Law*; that is to say, if the extensions in the principal directions are e_1, e_2, e_3 , and the corresponding stresses are R_1, R_2, R_3 , then

$$e_1 = \frac{1}{E} \left[R_1 - \frac{1}{m} (R_2 + R_3) \right], \dots, \&c.,$$

where E is YOUNG'S Modulus, and $\frac{1}{m}$ is POISSON'S ratio for the material under consideration.

These relations may be written in the form

$$\begin{aligned} R_1 &= \frac{mE}{(m+1)(m-2)} [(m-1)e_1 + e_2 + e_3] \\ &= \frac{2C}{m-2} [(m-1)e_1 + e_2 + e_3], \dots, \&c., \dots \dots \dots (2) \end{aligned}$$

where C is the Modulus of Rigidity.

In these relations the measure of extension is assumed to be

$$\frac{\text{Increase in length of linear element}}{\text{Length of the element before strain}}$$

and of stress*

$$\frac{\text{Total action over an element of surface}}{\text{Area of the element before strain}}$$

We have then the usual expression† for the energy of strain, per unit volume of the unstrained material, in terms of the principal extensions, viz.:—

$$W = \frac{m-1}{m-2} C (e_1 + e_2 + e_3)^2 - 2C (e_1 e_2 + e_2 e_3 + e_3 e_1). \dots \dots \dots (3)$$

The above assumptions yield sufficient data for the calculation of the stress system in any configuration of equilibrium, even when the strains are not small. Assuming that the calculation has been effected, we have to show how conditions for the stability of the system may be obtained.

We must distinguish three configurations: the unstrained configuration, in which the co-ordinates of any point are given by x, y, z ; the configuration of equilibrium under the stress-system, the stability of which we are investigating; and a configura-

* This assumption is open to the objection that it would render possible the compression of a material to zero volume by means of a finite stress. It will not, however, introduce any serious error, and has the advantage, which more probable assumptions do not possess, of leading to a definite energy-function. The definitions of stress and strain given above are generally employed in the construction of "stress-strain diagrams" from a tension test, the extensions of the specimen being taken as abscissæ, and the total loads as ordinates of the plotted curve.

† LOVE, *op. cit.*, § 68.

ration of slight distortion from the equilibrium position, which can be maintained without the introduction of additional stress at the boundaries, if the equilibrium of the second configuration is neutral. We shall consider first a stress-system which is such that the principal stresses in the second configuration have the same magnitudes and directions throughout the body;* and we shall take these directions as axes of x , y and z . We may then define the second and third configurations by saying that in them the co-ordinates of the point (x, y, z) become

$$\begin{aligned} & x(1+e_1), & y(1+e_2), & z(1+e_3), \\ \text{and} & x(1+e_1)+u', & y(1+e_2)+v', & z(1+e_3)+w', \end{aligned}$$

respectively. We shall not limit the values of e_1, e_2, e_3 , although in practical cases they must be small: u', v', w' are infinitesimal. In the second configuration the axes Ox, Oy, Oz are directions of principal stress, and the stresses are

$$X_x = \frac{2C}{m-2} [(m-1)e_1 + e_2 + e_3], \dots, \&c. \dots \dots \dots (2) \text{ bis}$$

In the third configuration we shall find that lines which in the first configuration were slightly inclined to Ox, Oy, Oz become directions of principal stress and strain. The final extension of a line which originally had direction-cosines l, m, n is

$$\begin{aligned} e' = -1 + \sqrt{\left[\left\{ l \left(1 + e_1 + \frac{\partial u'}{\partial x} \right) + m \frac{\partial u'}{\partial y} + n \frac{\partial u'}{\partial z} \right\}^2 \right.} \\ \left. + \left\{ l \frac{\partial v'}{\partial x} + m \left(1 + e_2 + \frac{\partial v'}{\partial y} \right) + n \frac{\partial v'}{\partial z} \right\}^2 + \left\{ l \frac{\partial w'}{\partial x} + m \frac{\partial w'}{\partial y} + n \left(1 + e_3 + \frac{\partial w'}{\partial z} \right) \right\}^2 \right]}. \dots (4) \end{aligned}$$

It may be shown that e' has a stationary value when

$$\left. \begin{aligned} l &= 1, \\ m &= m_1 = \frac{(1+e_1) \frac{\partial u'}{\partial y} + (1+e_2) \frac{\partial v'}{\partial x}}{(1+e_1)^2 - (1+e_2)^2}, \\ n &= n_1 = \frac{(1+e_1) \frac{\partial u'}{\partial z} + (1+e_3) \frac{\partial w'}{\partial x}}{(1+e_1)^2 - (1+e_3)^2}, \end{aligned} \right\} \dots \dots \dots (5)$$

and

to terms of the first order in u', v', w' .†

* In some cases, such as GREENHILL'S problem of the stability of a heavy vertical rod (p. 188, footnote), it is necessary to allow for variation in one or more of the principal stresses; the necessary alterations are easily made, and as they are not required for the examples of this paper their consideration would involve unnecessary complexity.

[† *Added May 1.*—The approximation of these expressions is insufficient if any two of the principal strains (e_1, e_2, e_3) in the second configuration are equal; in this case additional terms must be retained in the denominators. The equilibrium under hydrostatic stress ($e_1 = e_2 = e_3$) is necessarily and obviously stable.]

Thus the line initially given by the direction-cosines

$$1, \quad m_1, \quad n_1,$$

becomes a direction of principal stress in the final configuration. Its direction-cosines (referred to Ox , Oy , and Oz) are then

$$1, \quad \frac{\frac{\partial v'}{\partial x} + m_1(1+e_2)}{1+e_1}, \quad \frac{\frac{\partial w'}{\partial x} + n_1(1+e_3)}{1+e_1},$$

or

$$1, \quad \frac{(1+e_2)\frac{\partial u'}{\partial y} + (1+e_1)\frac{\partial v'}{\partial x}}{(1+e_1)^2 - (1+e_2)^2}, \quad \frac{(1+e_3)\frac{\partial u'}{\partial z} + (1+e_1)\frac{\partial w'}{\partial x}}{(1+e_1)^2 - (1+e_3)^2}, \quad \dots \quad (6)$$

which we shall write as $1, m'_1, n'_1$; and its final extension, to terms of the first order in u', v', w' , is

$$e'_1 = e_1 + \frac{\partial u'}{\partial x} \dots \dots \dots (7)$$

In the same way we find that the other directions of principal strain in the final configuration are given by the direction-cosines

$$\left. \begin{aligned} & \frac{(1+e_1)\frac{\partial v'}{\partial x} + (1+e_2)\frac{\partial u'}{\partial y}}{(1+e_2)^2 - (1+e_1)^2}, \quad 1, \quad \frac{(1+e_3)\frac{\partial v'}{\partial z} + (1+e_2)\frac{\partial w'}{\partial y}}{(1+e_2)^2 - (1+e_3)^2}, \\ & \frac{(1+e_1)\frac{\partial w'}{\partial x} + (1+e_3)\frac{\partial u'}{\partial z}}{(1+e_3)^2 - (1+e_1)^2}, \quad \frac{(1+e_2)\frac{\partial w'}{\partial y} + (1+e_3)\frac{\partial v'}{\partial z}}{(1+e_3)^2 - (1+e_2)^2}, \quad 1, \end{aligned} \right\} \dots \dots \dots (8)$$

(which we may write as $-m'_1, 1, n'_2$, and $-n'_1, -n'_2, 1$), and that the final extensions in these directions are

$$\left. \begin{aligned} & e'_2 = e_2 + \frac{\partial v'}{\partial y}, \\ & e'_3 = e_3 + \frac{\partial w'}{\partial z}. \end{aligned} \right\} \dots \dots \dots (9)$$

The stresses in these directions, which we shall call the directions of x', y' , and z' , referred to the *original* areas of the faces on which they act,* are therefore

$$\begin{aligned} X'_{x'} &= X_x + \frac{\partial X_x}{\partial e_1} \cdot \frac{\partial u'}{\partial x} + \frac{\partial X_x}{\partial e_2} \cdot \frac{\partial v'}{\partial y} + \frac{\partial X_x}{\partial e_3} \cdot \frac{\partial w'}{\partial z}, \\ &= X_x + \frac{2C}{m-2} \left[(m-1) \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} \right] \dots, \text{ \&c.} \quad \dots \quad (10) \end{aligned}$$

* Cf. the assumption of p. 193.

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Referred to the *new* areas of the faces on which they act, they are

$$\bar{X}'_{x'} = \frac{X'_{x'}}{(1+e'_2)(1+e'_3)}, \quad \bar{Y}'_{y'} = \frac{Y'_{y'}}{(1+e'_3)(1+e'_1)}, \quad \bar{Z}'_{z'} = \frac{Z'_{z'}}{(1+e'_1)(1+e'_2)},$$

and to the required degree of approximation we may write

$$\begin{aligned} \bar{X}'_{x'} = \frac{X_x}{(1+e_2)(1+e_3)} & \left[1 - \frac{1}{1+e_2} \cdot \frac{\partial v'}{\partial y} - \frac{1}{1+e_3} \cdot \frac{\partial w'}{\partial z} \right] \\ & + \frac{2C}{(1+e_2)(1+e_3)} \left[(m-1) \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} \right] \dots, \text{ \&c.} \quad \dots \quad (11) \end{aligned}$$

Then if \bar{x} , \bar{y} , \bar{z} denote the co-ordinates in the final configuration, referred to the original axes, of the point which was originally at (x, y, z) , so that

$$\bar{x} = x(1+e_1) + u', \quad \dots, \quad \text{\&c.},$$

we may find the stress components in the third configuration, *referred to the original axes, and to the strained areas of the faces upon which they act*, by the scheme of transformation

	x'	y'	z'
\bar{x}	1	$-m'_1$	$-n'_1$
\bar{y}	m'_1	1	$-n'_2$
\bar{z}	n'_1	n'_2	1

The following expressions are thus obtained (to the required order of approximation):—

$$\begin{aligned} \bar{X}_x &= \bar{X}'_{x'}, \quad \bar{Y}_y = \bar{Y}'_{y'}, \quad \bar{Z}_z = \bar{Z}'_{z'}, \\ \bar{X}_y &= m'_1 (\bar{X}'_{x'} - \bar{Y}'_{y'}), \\ &= m'_1 \left\{ \frac{(1+e_1)X_x - (1+e_2)Y_y}{(1+e_1)(1+e_2)(1+e_3)} \right\}, \dots, \text{ \&c.} \quad \dots \quad (12) \end{aligned}$$

Now the stress-components (12) must satisfy the ordinary equations of equilibrium, which are three of the type

$$\frac{\partial \bar{X}_x}{\partial x} + \frac{\partial \bar{X}_y}{\partial y} + \frac{\partial \bar{X}_z}{\partial z} = 0, \quad \dots \quad (13)$$

and since the co-ordinates of the point which ultimately goes to $(\bar{x} + \delta\bar{x}, \bar{y}, \bar{z})$ were originally

$$x + \frac{\delta\bar{x}}{1 + e_1 + \frac{\partial u'}{\partial x}}, \quad y - \frac{\frac{\partial v'}{\partial x} \cdot \delta\bar{x}}{(1 + e_1)(1 + e_2)}, \quad z - \frac{\frac{\partial w'}{\partial x} \cdot \delta\bar{x}}{(1 + e_3)(1 + e_1)},$$

we have

$$\frac{\partial}{\partial \bar{x}} \equiv \frac{1}{1 + e_1 + \frac{\partial u'}{\partial x}} \frac{\partial}{\partial x} - \frac{\frac{\partial v'}{\partial x}}{(1 + e_1)(1 + e_2)} \frac{\partial}{\partial y} - \frac{\frac{\partial w'}{\partial x}}{(1 + e_3)(1 + e_1)} \frac{\partial}{\partial z}, \dots, \text{ \&c.}$$

It follows that (13) may be written (to our approximation) as follows:—

$$\begin{aligned} & \frac{1}{(1 + e_1)(1 + e_2)(1 + e_3)} \left[-X_x \left\{ \frac{1}{1 + e_2} \frac{\partial^2 v'}{\partial x \partial y} + \frac{1}{1 + e_3} \frac{\partial^2 w'}{\partial z \partial x} \right\} \right. \\ & + \frac{2C}{m-2} \left\{ (m-1) \frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 v'}{\partial x \partial y} + \frac{\partial^2 w'}{\partial z \partial x} \right\} \\ & \left. + \frac{(1 + e_1) X_x - (1 + e_2) Y_y}{1 + e_2} \frac{\partial m'_1}{\partial y} + \frac{(1 + e_1) X_x - (1 + e_3) Z_z}{1 + e_3} \frac{\partial n'_1}{\partial z} \right] = 0. \quad (14) \end{aligned}$$

Substituting for m'_1, n'_1 , we have finally

$$\begin{aligned} & 2 \frac{m-1}{m-2} \frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2} + \frac{\partial^2 u'}{\partial z^2} + \frac{m}{m-2} \left(\frac{\partial^2 v'}{\partial x \partial y} + \frac{\partial^2 w'}{\partial z \partial x} \right) \\ & + \frac{X_x + Y_y}{4C \left(1 + \frac{e_1 + e_2}{2} \right)} \left(\frac{\partial^2 u'}{\partial y^2} - \frac{\partial^2 v'}{\partial x \partial y} \right) + \frac{X_x + Z_z}{4C \left(1 + \frac{e_1 + e_3}{2} \right)} \left(\frac{\partial^2 u'}{\partial z^2} - \frac{\partial^2 w'}{\partial z \partial x} \right) = 0, \quad (15) \end{aligned}$$

and two similar equations. In any ordinary problem we may neglect $e_1 \frac{X_x}{C} \dots$ in comparison with $\frac{X_x}{C} \dots$.

The equations thus obtained may also be written (with LAME'S notation for the elastic constants) as follows:—

$$(\lambda + \mu) \frac{\partial \Delta'}{\partial x} + \mu \nabla^2 u' - \frac{X_x + Y_y}{2 \left(1 + \frac{e_1 + e_2}{2} \right)} \frac{\partial \varpi'_z}{\partial y} + \frac{X_x + Z_z}{2 \left(1 + \frac{e_1 + e_3}{2} \right)} \frac{\partial \varpi'_y}{\partial z} = 0, \dots, \text{ \&c.}, \quad (16)$$

where

$$\begin{aligned} \Delta' &= \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}, \\ \nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \\ 2\varpi'_x &= \frac{\partial w'}{\partial y} - \frac{\partial v'}{\partial z}, \quad 2\varpi'_y = \frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x}, \quad 2\varpi'_z = \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}, \end{aligned}$$

and in this form they may be conveniently compared with the ordinary equations of elasticity.*

The three equations of the type (15) we shall term *Equations of Neutral Equilibrium*. The equilibrium of the stress-system X_x, Y_y, Z_z will be neutral, provided that solutions for u', v', w' exist which satisfy certain boundary conditions. These boundary conditions are peculiar to each problem, but usually express the condition that the additional stresses involved by u', v', w' shall vanish on certain boundary surfaces. *They never determine the magnitude of u', v', w'* , so that our solution gives the *form* only of the distortion which tends to occur in the body under consideration when its equilibrium becomes unstable. It gives a definite relation between the stress-system $X_x \dots$ and the dimensions of the body, which must be satisfied in order that *any* distortion may be permanent; but if this relation be satisfied, no limits are imposed by the equations upon the magnitude of the distortion which may occur.†

Example in Rectangular Co-ordinates. Stability of Thin Plating under Edge Thrust.

It seems advisable, before we employ a new method on problems which have not as yet received satisfactory treatment, in some degree to test its validity by the result to which it leads in a more familiar example. For this purpose we may consider the stability of an infinite strip of flat plating under edge thrusts in its plane. The accepted formula‡ for the thrust necessary to produce instability, per unit length of edge, is

$$\mathfrak{B} = \frac{2}{3} \frac{m^2}{m^2 - 1} E \pi^2 \frac{t^3}{l^2}, \dots \dots \dots (17)$$

where

$2t$ = thickness of plate,

l = breadth of plate,

and the opposite edges are simply supported. If the edges are built in, the thrust required has four times this value.

To investigate this problem by the new method we take axes Ox and Oz in the middle surface of the plate, in the direction of its breadth and length respectively, and Oy perpendicular to the middle surface. The initial stress-system is then given by

$$\left. \begin{aligned} X_x &= \text{const.} = G \text{ (say),} \\ Y_y &= Z_z = 0; \end{aligned} \right\} \dots \dots \dots (18)$$

* LOVE, *op. cit.*, § 91, equation (19).

† The equations are, however, rigorous only in the case of *infinitesimal* displacements; *cf.* footnote, p. 240.

‡ *Cf.* LOVE, *op. cit.*, § 337 (a), whence the above expression may be obtained.

and we may assume that the system of strain which is introduced at collapse will be two-dimensional, so that

$$\frac{\partial u'}{\partial z} = \frac{\partial v'}{\partial z} = 0, \quad \frac{w'}{z} = \text{const.} \quad \dots \quad (19)$$

The third equation of neutral stability (for the direction Oz) is then satisfied identically, and the other two equations become (if we neglect terms of order $\frac{G^2}{C^2} u' \dots$)

$$\text{and} \quad \left. \begin{aligned} 2 \frac{m-1}{m-2} \cdot \frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2} + \frac{m}{m-2} \cdot \frac{\partial^2 v'}{\partial x \partial y} - \frac{G}{4C} \frac{\partial}{\partial y} \left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} \right) &= 0, \\ \frac{\partial^2 v'}{\partial x^2} + 2 \frac{m-1}{m-2} \cdot \frac{\partial^2 v'}{\partial y^2} + \frac{m}{m-2} \cdot \frac{\partial^2 u'}{\partial x \partial y} - \frac{G}{4C} \frac{\partial}{\partial x} \left(\frac{\partial u'}{\partial y} - \frac{\partial v'}{\partial x} \right) &= 0. \end{aligned} \right\} \dots \quad (20)$$

Let us assume a solution of the form

$$\left. \begin{aligned} u' &= \Sigma [U_a \sin \alpha (x+x_0)] \\ v' &= \Sigma [V_a \cos \alpha (x+x_0)] \end{aligned} \right\}, \quad \dots \quad (21)$$

where U_a and V_a are functions of y only. It is easy to show that this assumption as to the phase-relation of u' and v' is justified. We have then

$$\text{and} \quad \left. \begin{aligned} -2 \frac{m-1}{m-2} \alpha^2 U_a + \left(1 + \frac{G}{4C} \right) \frac{d^2 U_a}{dy^2} - \left(\frac{m}{m-2} - \frac{G}{4C} \right) \alpha \frac{dV_a}{dy} &= 0, \\ - \left(1 + \frac{G}{4C} \right) \alpha^2 V_a + 2 \frac{m-1}{m-2} \cdot \frac{d^2 V_a}{dy^2} + \left(\frac{m}{m-2} - \frac{G}{4C} \right) \alpha \frac{dU_a}{dy} &= 0. \end{aligned} \right\} \dots \quad (22)$$

The solution of these equations is of the form

$$\left. \begin{aligned} U_a &= (Py+Q) \sinh \alpha y + (Ry+S) \cosh \alpha y, \\ V_a &= - \left\{ Ry+S - \left(\frac{3m-4}{m-2} + \frac{G}{4C} \right) \frac{P}{\alpha} \right\} \sinh \alpha y \\ &\quad - \left\{ Py+Q - \left(\frac{3m-4}{m-2} - \frac{G}{4C} \right) \frac{R}{\alpha} \right\} \cosh \alpha y, \end{aligned} \right\} \dots \quad (23)$$

where P, Q, R, S are constants.

The boundary conditions now demand attention. It is clear that the stresses introduced by u', v', w' must vanish at the surfaces of the plate. Hence these surfaces will still be planes of principal stress, and, moreover, the normal stress upon

them must vanish. But, as we have already seen, the line which becomes a direction Oy' of principal stress has initially the direction-cosines

$$-m_1, \quad 1, \quad n_2;$$

it follows that at the surfaces of the plate the expression for m_1 and n_2 must vanish identically; moreover, at these surfaces, Y'_y must vanish. These conditions may be written in the form

$$\left. \begin{aligned} (1+e_2) \frac{\partial v'}{\partial z} + (1+e_3) \frac{\partial w'}{\partial y} &= 0, \\ (1+e_1) \frac{\partial u'}{\partial y} + (1+e_2) \frac{\partial v'}{\partial x} &= 0, \\ (m-1) \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} + \frac{\partial u'}{\partial x} &= 0, \end{aligned} \right\} \text{identically, when } y = \pm t. \quad (24)$$

The first condition is already satisfied. The other two give (if we neglect terms of order $\frac{G^2}{C^2} u' \dots$)

$$\left. \begin{aligned} (1+e_1) \frac{dU_a}{dy} - (1+e_2) \alpha V_a &= 0, \\ (m-1) \frac{dV_a}{dy} + \alpha U_a &= 0, \end{aligned} \right\} \text{when } y = \pm t, \dots \quad (25)$$

or

$$\begin{aligned} & \text{P} \left[\left(1 + \frac{m-1}{m+1} \cdot \frac{G}{4C} \right) \alpha y \cosh \alpha y - \left(\frac{1 - \frac{2m-1}{(m+1)(m-2)} \cdot \frac{G}{2C}}{\frac{m}{m-2} - \frac{G}{4C}} \right) \sinh \alpha y \right] \\ & \quad + \text{Q} \left(1 + \frac{m-1}{m+1} \cdot \frac{G}{4C} \right) \alpha \cosh \alpha y \\ & + \text{R} \left[\left(1 + \frac{m-1}{m+1} \cdot \frac{G}{4C} \right) \alpha y \sinh \alpha y - \left(\frac{1 - \frac{2m-1}{(m+1)(m-2)} \cdot \frac{G}{2C}}{\frac{m}{m-2} - \frac{G}{4C}} \right) \cosh \alpha y \right] \\ & \quad + \text{S} \left(1 + \frac{m-1}{m+1} \cdot \frac{G}{4C} \right) \alpha \sinh \alpha y = 0, \dots \quad (26) \end{aligned}$$

and

$$\begin{aligned} & \text{P} \left[(m-2) \alpha y \sinh \alpha y - 2(m-1) \frac{1 + \frac{G}{4C}}{\frac{m}{m-2} - \frac{G}{4C}} \cosh \alpha y \right] + \text{Q} (m-2) \alpha \sinh \alpha y \\ & + \text{R} \left[(m-2) \alpha y \cosh \alpha y - 2(m-1) \frac{1 + \frac{G}{4C}}{\frac{m}{m-2} - \frac{G}{4C}} \sinh \alpha y \right] + \text{S} (m-2) \alpha \cosh \alpha y = 0 \end{aligned}$$

when

$$y = \pm t \dots \quad (27)$$

Thus we obtain

$$\left. \begin{aligned} \alpha S &= P \left[\frac{1 - \frac{2m-1}{(m+1)(m-2)} \cdot \frac{G}{2C}}{\left(\frac{m}{m-2} - \frac{G}{4C}\right) \left(1 + \frac{m-1}{m+1} \cdot \frac{G}{4C}\right)} - at \coth at \right], \\ \alpha Q &= R \left[\frac{1 - \frac{2m-1}{(m+1)(m-2)} \cdot \frac{G}{2C}}{\left(\frac{m}{m-2} - \frac{G}{4C}\right) \left(1 + \frac{m-1}{m+1} \cdot \frac{G}{4C}\right)} - at \tanh at \right], \\ \alpha S &= P \left[2 \frac{m-1}{m-2} \left(\frac{1 + \frac{G}{4C}}{\frac{m}{m-2} - \frac{G}{4C}} \right) - at \tanh at \right], \\ \alpha Q &= R \left[2 \frac{m-1}{m-2} \left(\frac{1 + \frac{G}{4C}}{\frac{m}{m-2} - \frac{G}{4C}} \right) - at \coth at \right]. \end{aligned} \right\} \dots (28)$$

There are two solutions of the equations (28). Either

$$\frac{1 - \frac{2m-1}{(m+1)(m-2)} \cdot \frac{G}{2C}}{\left(\frac{m}{m-2} - \frac{G}{4C}\right) \left(1 + \frac{m-1}{m+1} \cdot \frac{G}{4C}\right)} - at \tanh at = \frac{\alpha Q}{R} = 2 \frac{m-1}{m-2} \left(\frac{1 + \frac{G}{4C}}{\frac{m}{m-2} - \frac{G}{4C}} \right) - at \coth at,$$

and

$$P = S = 0, \dots (29)$$

or

$$\frac{1 - \frac{2m-1}{(m+1)(m-2)} \cdot \frac{G}{2C}}{\left(\frac{m}{m-2} - \frac{G}{4C}\right) \left(1 + \frac{m-1}{m+1} \cdot \frac{G}{4C}\right)} - at \coth at = \frac{\alpha S}{P} = 2 \frac{m-1}{m-2} \left(\frac{1 + \frac{G}{4C}}{\frac{m}{m-2} - \frac{G}{4C}} \right) - at \tanh at,$$

and

$$Q = R = 0. \dots (30)$$

The criterion for neutral stability is in the first case

$$at (\coth at - \tanh at) = \left(\frac{1 + \frac{2m^2-1}{2m(m+1)} \cdot \frac{G}{C}}{1 + \frac{1}{2m(m+1)} \cdot \frac{G}{C}} \right),$$

and in the second case

$$at (\tanh at - \coth at) = \left(\frac{1 + \frac{2m^2-1}{2m(m+1)} \cdot \frac{G}{C}}{1 + \frac{1}{2m(m+1)} \cdot \frac{G}{C}} \right),$$

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so that the values of G , for which collapse by instability may be expected to occur, are given by

$$-\frac{G}{2m(m+1)C} = \frac{1-2at \operatorname{cosech} 2at}{2m^2-1-2at \operatorname{cosech} 2at} \dots \dots \dots (31)$$

and

$$-\frac{G}{2m(m+1)C} = \frac{1+2at \operatorname{cosech} 2at}{2m^2-1+2at \operatorname{cosech} 2at} \dots \dots \dots (32)$$

respectively, the total thrust, per unit length of edge, being

$$\mathfrak{B} = -2tG \dots \dots \dots (33)$$

The first approximations to a solution, in terms of t , are

$$G = -\frac{2}{3} \cdot \frac{mC}{m-1} \cdot \alpha^2 t^2 \dots \dots \dots (34)$$

and

$$G = -2 \frac{m+1}{m} C = -E \dots \dots \dots (35)$$

respectively. Since the complete wave-length of the corrugations into which the plate distorts is

$$\lambda = \frac{2\pi}{\alpha}, \dots \dots \dots (36)$$

we see that (34) is equivalent to (17), and that the latter formula is therefore supported by our investigation *as a first approximation*. The second solution (35) is without practical interest, owing to the magnitude of the thrust required to produce collapse. It refers to a type of distortion, theoretically possible for an ideal material without limits of elasticity, which is approximately realized in actual specimens of ductile material, when tested to failure under compressive stress. Since $Q = R = 0$, we see from (23) that in this type *the middle surface remains plane*. In the first type of failure, where $P = S = 0$, we find that $U_a = 0$ when $y = 0$, so that *the middle surface undergoes no change of extension* in the distortion given by u', v', w' .*

EQUATIONS OF NEUTRAL EQUILIBRIUM IN CYLINDRICAL CO-ORDINATES.

Derivation of the Equations.

The equations (15) of neutral equilibrium are expressed in a form which is unsuitable for the investigation of problems concerned with the stability of thin tubes, and we have next to obtain the corresponding equations in cylindrical

* Besides the harmonic solutions to (20) we may have

$$u' = gx, \quad v' = hy, \quad w' = kz;$$

but g, h , and k vanish in virtue of the boundary conditions.

co-ordinates. We shall limit our discussion to stress-systems which produce a displacement symmetrical about an axis, up to the instant at which the equilibrium becomes unstable and distortion occurs: in PEARSON'S notation, the principal stresses in the equilibrium configuration are \widehat{rr} , $\widehat{\theta\theta}$, and \widehat{zz} , and these quantities are functions of r only.

The new equations are derived by a method very similar to that which has already been explained. The co-ordinates of a point in the unstrained configuration are

$$r, \theta, z;$$

in the second configuration (of equilibrium) they are

$$r+u, \theta, z+w,$$

and in the third configuration (of slight distortion from the position of equilibrium) they are

$$r+u+u', \quad \theta + \frac{v'}{r+u}, \quad z+w+w',$$

(the radial, tangential, and axial displacements u' , v' , w' being ultimately taken as infinitesimal).

The extension of a line-element joining the point (r, θ, z) to the point $(r+\delta r, \theta+\delta\theta, z+\delta z)$ is

$$\begin{aligned} e' = -1 + \sqrt{\left[\left(\frac{1}{1+m^2+n^2} \right) \left\{ \left[1 + \frac{\partial u}{\partial r} + \frac{\partial u'}{\partial r} + m \left(\frac{1}{r} \frac{\partial u'}{\partial \theta} - \frac{v'}{r} \right) + n \frac{\partial u'}{\partial z} \right]^2 \right. \right.} \\ \left. \left. + \left[\frac{\partial v'}{\partial r} + m \left(1 + \frac{u}{r} + \frac{u'}{r} + \frac{1}{r} \frac{\partial v'}{\partial \theta} \right) + n \frac{\partial v'}{\partial z} \right]^2 \right. \right.} \\ \left. \left. + \left[\frac{\partial w'}{\partial r} + \frac{m}{r} \frac{\partial w'}{\partial \theta} + n \left(1 + \frac{\partial w}{\partial z} + \frac{\partial w'}{\partial z} \right) \right]^2 \right\} \right]}, \dots \dots \dots (37) \end{aligned}$$

where

$$m = r \frac{\delta\theta}{\delta r}, \quad \text{and} \quad n = \frac{\delta z}{\delta r};$$

and this has a stationary value for a line very slightly inclined to the radius, given by

$$\left. \begin{aligned} m_1 &= \frac{(1+e_1) \left(\frac{1}{r} \frac{\partial u'}{\partial \theta} - \frac{v'}{r} \right) + (1+e_2) \frac{\partial v'}{\partial r}}{(1+e_1)^2 - (1+e_2)^2}, \\ n_1 &= \frac{(1+e_1) \frac{\partial u'}{\partial z} + (1+e_3) \frac{\partial w'}{\partial r}}{(1+e_1)^2 - (1+e_3)^2}, \end{aligned} \right\} \dots \dots \dots (38)$$

where e_1 , e_2 , e_3 are written for $\partial u/\partial r$, u/r , and $\partial w/\partial z$ respectively.

The extension of this line-element in the third configuration is

$$e_1' = e_1 + \frac{\partial u'}{\partial r}, \quad \dots \dots \dots (39)$$

and its inclination to lines issuing from the point (in its final position) in the radial, tangential, and axial directions is given by the direction cosines

$$1, \quad m'_1, \quad n'_1,$$

where

$$\left. \begin{aligned} m'_1 &= \frac{(1+e_2)\left(\frac{1}{r} \frac{\partial u'}{\partial \theta} - \frac{v'}{r}\right) + (1+e_1) \frac{\partial v'}{\partial r} - \frac{v'}{r}}{(1+e_1)^2 - (1+e_2)^2}, \\ n'_1 &= \frac{(1+e_3) \frac{\partial u'}{\partial z} + (1+e_1) \frac{\partial w'}{\partial r}}{(1+e_1)^2 - (1+e_3)^2}. \end{aligned} \right\} \dots \dots \dots (40)$$

We find also that the other directions of principal stress in the final configuration are initially inclined to radial, tangential, and axial lines through (r, θ, z) at angles whose direction cosines are

$$-m_1, \quad 1, \quad n_2,$$

and

$$-n_1, \quad -n_2, \quad 1,$$

and that in the third configuration they have corresponding inclinations to radial, tangential, and axial lines through (r, θ, z) —in its final position—which are given by

$$-m'_1, \quad 1, \quad n'_2,$$

and

$$-n'_1, \quad -n'_2, \quad 1,$$

where

$$\left. \begin{aligned} n_2 &= \frac{(1+e_2) \frac{\partial v'}{\partial z} + (1+e_3) \frac{1}{r} \frac{\partial w'}{\partial \theta}}{(1+e_2)^2 - (1+e_3)^2}, \\ n'_2 &= \frac{(1+e_3) \frac{\partial v'}{\partial z} + (1+e_2) \frac{1}{r} \frac{\partial w'}{\partial \theta}}{(1+e_2)^2 - (1+e_3)^2}. \end{aligned} \right\} \dots \dots \dots (41)$$

The extensions of the corresponding line-elements in the final configuration are respectively

$$\left. \begin{aligned} e'_2 &= e_2 + \frac{u'}{r} + \frac{1}{r} \frac{\partial v'}{\partial \theta}, \\ e'_3 &= e_3 + \frac{\partial w'}{\partial z}, \end{aligned} \right\} \dots \dots \dots (42)$$

and

so that the principal stresses in the third configuration, referred to the *final* areas of the faces on which they act, are

$$\left. \begin{aligned}
 \widehat{rr}' &= \frac{\widehat{rr}}{(1+e_2)(1+e_3)} \left[1 - \frac{u'}{r} + \frac{1}{r} \frac{\partial v'}{\partial \theta} - \frac{\partial w'}{\partial z} \right] \\
 &\quad + \frac{2C}{(1+e_2)(1+e_3)} \frac{m-2}{m-2} \left[(m-1) \frac{\partial u'}{\partial r} + \frac{u'}{r} + \frac{1}{r} \frac{\partial v'}{\partial \theta} + \frac{\partial w'}{\partial z} \right], \\
 \widehat{\theta\theta}' &= \frac{\widehat{\theta\theta}}{(1+e_3)(1+e_1)} \left[1 - \frac{\partial w'}{\partial z} - \frac{\partial u'}{\partial r} \right] \\
 &\quad + \frac{2C}{(1+e_3)(1+e_1)} \frac{m-2}{m-2} \left[(m-1) \left(\frac{u'}{r} + \frac{1}{r} \frac{\partial v'}{\partial \theta} \right) + \frac{\partial w'}{\partial z} + \frac{\partial u'}{\partial r} \right], \\
 \widehat{zz}' &= \frac{\widehat{zz}}{(1+e_1)(1+e_2)} \left[1 - \frac{\partial u'}{\partial r} - \frac{u'}{r} + \frac{1}{r} \frac{\partial v'}{\partial \theta} \right] \\
 &\quad + \frac{2C}{(1+e_1)(1+e_2)} \frac{m-2}{m-2} \left[(m-1) \frac{\partial w'}{\partial z} + \frac{\partial u'}{\partial r} + \frac{u'}{r} + \frac{1}{r} \frac{\partial v'}{\partial \theta} \right].
 \end{aligned} \right\} \dots (43)$$

Then by the scheme of transformation

	r'	θ'	z'
\bar{r}	1	$-m'_1$	$-n'_1$
$\bar{\theta}$	m'_1	1	$-n'_2$
\bar{z}	n'_1	n'_2	1

we find for the stress components in the third configuration, at the point which was initially at (r, θ, z) ,—referred to axes in the radial, tangential, and axial directions through the *final* position of the point, and to the *final* areas of the faces upon which they act—the expressions

$$\begin{aligned}
 \widehat{rr} &= \widehat{rr}', & \widehat{\theta\theta} &= \widehat{\theta\theta}', & \widehat{zz} &= \widehat{zz}', \\
 \widehat{r\theta} &= \widehat{\theta r} = m'_1 (\widehat{rr}' - \widehat{\theta\theta}') = \frac{m'_1}{(1+e_1)(1+e_2)(1+e_3)} [(1+e_1)\widehat{rr} - (1+e_2)\widehat{\theta\theta}] \text{ (to the} \\
 &\quad \text{required approximation), \&c.} \dots \dots \dots (44)
 \end{aligned}$$

The equations of equilibrium, to be satisfied by the stress components (44), are*

$$\left. \begin{aligned} \frac{\widehat{\partial rr}}{\partial r} + \frac{1}{r} \frac{\widehat{\partial r\theta}}{\partial \theta} + \frac{\widehat{\partial zr}}{\partial z} + \frac{r\theta - \widehat{\theta\theta}}{r} &= 0, \\ \frac{\widehat{\partial r\theta}}{\partial r} + \frac{1}{r} \frac{\widehat{\partial \theta\theta}}{\partial \theta} + \frac{\widehat{\partial \theta z}}{\partial z} + 2 \frac{r\theta}{r} &= 0, \\ \frac{\widehat{\partial zr}}{\partial r} + \frac{1}{r} \frac{\widehat{\partial \theta z}}{\partial \theta} + \frac{\widehat{\partial zz}}{\partial z} + \frac{zr}{r} &= 0, \end{aligned} \right\} \dots \dots \dots (45)$$

where

$$r = r + u + u',$$

$$\theta = \theta + \frac{v'}{r + u},$$

and

$$z = z + w + w'.$$

Moreover, since

$$\frac{\partial}{\partial r} = \frac{1}{1 + e_1 + \frac{\partial u'}{\partial r}} \cdot \frac{\partial}{\partial r} - \frac{\frac{\partial v'}{\partial r} - \frac{1 + e_1}{1 + e_2} \cdot \frac{v'}{r}}{(1 + e_1)(1 + e_2)} \cdot \frac{1}{r} \frac{\partial}{\partial \theta} - \frac{\frac{\partial w'}{\partial r}}{(1 + e_3)(1 + e_1)} \cdot \frac{\partial}{\partial z},$$

$$\frac{1}{r} \cdot \frac{\partial}{\partial \theta} = \frac{-\frac{1}{r} \frac{\partial u'}{\partial \theta}}{(1 + e_1)(1 + e_2)} \cdot \frac{\partial}{\partial r} + \frac{1}{1 + e_2 + \frac{u'}{r} + \frac{1}{r} \frac{\partial v'}{\partial \theta}} \cdot \frac{1}{r} \frac{\partial}{\partial \theta} - \frac{\frac{1}{r} \frac{\partial w'}{\partial \theta}}{(1 + e_2)(1 + e_3)} \cdot \frac{\partial}{\partial z},$$

and

$$\frac{\partial}{\partial z} = \frac{-\frac{\partial u'}{\partial z}}{(1 + e_3)(1 + e_1)} \cdot \frac{\partial}{\partial r} - \frac{\frac{\partial v'}{\partial z}}{(1 + e_2)(1 + e_3)} \cdot \frac{1}{r} \frac{\partial}{\partial \theta} + \frac{1}{1 + e_3 + \frac{\partial w'}{\partial z}} \cdot \frac{\partial}{\partial z},$$

the equations (45) may be expressed in differentials with respect to r , θ , z . The terms which do not involve u' , v' , w' vanish in virtue of the equilibrium conditions for the second configuration, and only terms of the first order in these quantities need be retained.

In general, e_1 , e_2 , e_3 may all be functions of r , but in this paper we shall only consider problems in which e_3 and \widehat{zz} have constant values. Moreover, in all problems of practical importance we may neglect terms of order $\frac{r^2}{C^2} u' \dots$ in comparison with

* LOVE, *op. cit.*, § 59 (i).

terms of order $\frac{\widehat{rr}}{C} u' \dots$. We then obtain, as the equations of neutral equilibrium in cylindrical co-ordinates,

$$2 \frac{m-1}{m-2} \left(\frac{\partial^2 u'}{\partial r^2} + \frac{1}{r} \frac{\partial u'}{\partial r} - \frac{u'}{r^2} \right) + \frac{1}{r^2} \frac{\partial^2 u'}{\partial \theta^2} + \frac{\partial^2 u'}{\partial z^2} - \frac{3m-4}{m-2} \cdot \frac{1}{r^2} \frac{\partial v'}{\partial \theta} + \frac{m}{m-2} \left(\frac{1}{r} \frac{\partial^2 v'}{\partial r \partial \theta} + \frac{\partial^2 w'}{\partial z \partial r} \right) \\ + \left(\frac{\widehat{rr} + \widehat{\theta\theta}}{4C} \right) \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial u'}{\partial \theta} - \frac{\partial v'}{\partial r} - \frac{v'}{r} \right) + \left(\frac{\widehat{rr} + \widehat{zz}}{4C} \right) \frac{\partial}{\partial z} \left(\frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial r} \right) = 0, \quad \dots \quad (46)$$

$$\frac{m}{m-2} \cdot \frac{1}{r} \frac{\partial^2 u'}{\partial r \partial \theta} + \frac{3m-4}{m-2} \cdot \frac{1}{r^2} \frac{\partial u'}{\partial \theta} + \frac{\partial^2 v'}{\partial r^2} + \frac{1}{r} \frac{\partial v'}{\partial r} - \frac{v'}{r^2} + 2 \frac{m-1}{m-2} \cdot \frac{1}{r^2} \frac{\partial^2 v'}{\partial \theta^2} + \frac{\partial^2 v'}{\partial z^2} + \frac{m}{m-2} \cdot \frac{1}{r} \frac{\partial^2 w'}{\partial \theta \partial z} \\ + \left(\frac{\widehat{\theta\theta} + \widehat{zz}}{4C} \right) \frac{\partial}{\partial z} \left(\frac{\partial v'}{\partial z} - \frac{1}{r} \frac{\partial w'}{\partial \theta} \right) + \frac{\partial}{\partial r} \left[\left(\frac{\widehat{rr} + \widehat{\theta\theta}}{4C} \right) \left(\frac{\partial v'}{\partial r} + \frac{v'}{r} - \frac{1}{r} \frac{\partial u'}{\partial \theta} \right) \right] = 0, \quad \dots \quad (47)$$

and

$$\frac{m}{m-2} \left(\frac{\partial^2 u'}{\partial z \partial r} + \frac{1}{r} \frac{\partial u'}{\partial z} + \frac{1}{r} \frac{\partial^2 v'}{\partial \theta \partial z} \right) + \frac{\partial^2 w'}{\partial r^2} + \frac{1}{r} \frac{\partial w'}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w'}{\partial \theta^2} + 2 \frac{m-1}{m-2} \cdot \frac{\partial^2 w'}{\partial z^2} \\ + \left(\frac{\widehat{\theta\theta} + \widehat{zz}}{4C} \right) \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial w'}{\partial \theta} - \frac{\partial v'}{\partial z} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left[r \left(\frac{\widehat{rr} + \widehat{zz}}{4C} \right) \left(\frac{\partial w'}{\partial r} - \frac{\partial u'}{\partial z} \right) \right] = 0. \quad \dots \quad (48)$$

Equations (46–48) represent the conditions for neutral stability in the equilibrium of a body subjected to a stress-system \widehat{rr} , $\widehat{\theta\theta}$, \widehat{zz} , where \widehat{zz} is constant, and \widehat{rr} and $\widehat{\theta\theta}$ are functions of r only, which satisfy the condition of equilibrium

$$\frac{\partial \widehat{rr}}{\partial r} + \frac{\widehat{rr} - \widehat{\theta\theta}}{r} = 0. \quad \dots \quad (49)$$

For comparison with the ordinary equations of equilibrium in cylindrical co-ordinates they may be written (with LAMÉ'S notation for the elastic constants) in the forms*

$$\left. \begin{aligned} (\lambda + 2\mu) \frac{\partial \Delta'}{\partial r} - \frac{1}{r} \frac{\partial}{\partial \theta} \left[\left(2\mu + \frac{\widehat{rr} + \widehat{\theta\theta}}{2} \right) \varpi'_z \right] + \frac{\partial}{\partial z} \left[\left(2\mu + \frac{\widehat{rr} + \widehat{zz}}{2} \right) \varpi'_\theta \right] &= 0, \\ (\lambda + 2\mu) \frac{1}{r} \frac{\partial \Delta'}{\partial \theta} - \frac{\partial}{\partial z} \left[\left(2\mu + \frac{\widehat{\theta\theta} + \widehat{zz}}{2} \right) \varpi'_r \right] + \frac{\partial}{\partial r} \left[\left(2\mu + \frac{\widehat{rr} + \widehat{\theta\theta}}{2} \right) \varpi'_z \right] &= 0, \end{aligned} \right\} \quad (50)$$

and

$$(\lambda + 2\mu) \frac{\partial \Delta'}{\partial z} - \frac{1}{r} \frac{\partial}{\partial r} \left[r \varpi'_\theta \left(2\mu + \frac{\widehat{rr} + \widehat{zz}}{2} \right) \right] + \frac{1}{r} \frac{\partial}{\partial \theta} \left[\left(2\mu + \frac{\widehat{\theta\theta} + \widehat{zz}}{2} \right) \varpi'_r \right] = 0,$$

where

$$\Delta' = \frac{1}{r} \frac{\partial}{\partial r} (ru') + \frac{1}{r} \frac{\partial v'}{\partial \theta} + \frac{\partial w'}{\partial z},$$

* Cf. LOVE, *op. cit.*, § 199.

and

$$2\varpi'_r = \frac{1}{r} \frac{\partial w'}{\partial \theta} - \frac{\partial v'}{\partial z}, \quad 2\varpi'_\theta = \frac{\partial w'}{\partial z} - \frac{\partial v'}{\partial r}, \quad 2\varpi'_z = \frac{1}{r} \left\{ \frac{\partial}{\partial r} (rv') - \frac{\partial u'}{\partial \theta} \right\},$$

so that

$$\frac{1}{r} \frac{\partial}{\partial r} (r\varpi'_r) + \frac{1}{r} \frac{\partial \varpi'_\theta}{\partial \theta} + \frac{\partial \varpi'_z}{\partial z} = 0.$$

Examples in Cylindrical Co-ordinates. Stability of Boiler Flues and Tubular Struts.

The Equations of Neutral Equilibrium in Cylindrical Co-ordinates enable us to deal successfully with some difficult problems connected with the stability of cylindrical tubes. Two examples of considerable importance will be discussed in this paper—the collapse of boiler flues and the strength of tubular struts. It should be noticed that neither of these problems has been quite satisfactorily treated by the ordinary theory



Fig. 2.

of thin shells, which requires the assumptions that the middle surface of the shell is unextended, and the inner and outer surfaces free from applied tractions*; hence their solution is a problem of considerable interest, even apart from practical considerations, and has attracted a great deal of attention. It will be convenient at this point to review the work which has already been done.

The question of the stability of tubular struts is important, owing to the frequency of their employment in practice. In economy of material the cylindrical tube possesses an advantage over struts of solid cross-section, and both the theory of

* Cf. A. B. BASSET, 'Phil. Trans. Roy. Soc.,' A, CLXXXI., p. 437; and RAYLEIGH, 'London Math. Soc. Proc.,' vol. XX., p. 379.

EULER* and LAGRANGE† and the more practical formula of RANKINE suggest that this advantage increases without limit as the thickness of the tube is reduced. Such a conclusion is, however, inaccurate, for types of distortion are possible in the case of a tube which do not involve flexure of the axis, and when the tube is thin these types, of which some practical examples are shown in fig. 2, may be maintained by a smaller thrust than would be required to produce failure of the kind discussed by EULER. Moreover, the natural wave-length, for these symmetrical types of distortion, is in general small, so that distortion can occur without hindrance in quite short tubes. Hence, for a considerable range of length the strength of a tube to resist end-thrust is practically constant, and is not given by any of the usual formulæ for struts.

The determination of the strength of tubes to resist these symmetrical types of distortion is obviously a problem of the highest practical importance, and has attracted a great deal of attention in recent years. Illustrations of collapsed tubes, showing symmetrical types of distortion, have been published by A. MALLOCK‡ and R. LORENZ,§ and a great deal of experimental work has been carried out by W. E. LILLY.|| Theoretical discussions, by approximate methods, have been proposed by A. GROS,¶ W. E. LILLY,** S. TIMOSCHENKO†† and R. LORENZ.‡‡

The problem of the boiler flue seems first to have been suggested by the experiments commenced by FAIRBAIRN in 1858.§§ These showed that the collapse of tubes under external pressure was in some degree analogous to that of straight columns under end-thrust, and a discussion of the phenomenon, based on EULER'S theory of struts, was given by W. C. UNWIN,||| who assisted FAIRBAIRN in his research. The similar problem of a circular wire ring subjected to radial pressure has been discussed by M. BRESSE¶¶ and M. LÉVY,*** and rational theories of the boiler-flue problem have been given by G. H. BRYAN,††† A. FÖPPL,‡‡‡ P. FORCHHEIMER§§§

* "Sur la force des colonnes," 'Hist. Acad. Berlin,' XIII. (1757), p. 252.

† "Sur la figure des colonnes," 'Miscellanea Taurinensia,' V. (1773).

‡ 'Roy. Soc. Proc.,' A, LXXXI. (1908), p. 389.

§ 'Physikalische Zeitschrift,' XII. (1911), p. 241.

|| 'Proc. Inst. Mech. Eng.,' 1905; 'Inst. Civ. Eng. Ireland,' 1906; 'Engineering,' January 10, 1908.

¶ 'Comptes Rendus,' CXXXIV. (1902), p. 1041.

** 'Inst. Civ. Eng. Ireland,' 1906.

†† 'Zeitschrift f. Mathematik u. Physik,' LVIII. (1910), p. 337.

‡‡ 'Zeitschrift des Vereines Deutscher Ingenieure,' October 24, 1908; 'Physikalische Zeitschrift,' XII. (1911), p. 241.

§§ 'Phil. Trans. Roy. Soc.,' CXLVIII., p. 389.

||| 'Proc. Inst. C.E.,' XLVI. (1875), p. 225.

¶¶ 'Cours de Mécanique Appliquée,' 1. Partie, Paris, 1859.

*** 'LIOUVILLE'S Journal,' X. (1884), p. 5.

††† 'Proc. Camb. Phil. Soc.,' VI. (1888), p. 287.

‡‡‡ 'Résistance des Matériaux' (1901), p. 286.

§§§ 'Zeitschrift des Oesterreichischen Ingenieur- und Architekten-Vereines,' 1904.

and R. LORENZ.* W. E. LILLY† has indicated the correct form of the result for an infinitely long flue, and A. E. H. LOVE‡ has discussed the strengthening effects of constraints which keep the tube circular at its ends.

A. B. BASSET§ has given a very clear exposition of the difficulties which are encountered in an attempt to construct a theory of flue collapse by usual methods. To obtain sufficient equations we must assume that the middle surface undergoes no extension; and the existence of pressure on one or both surfaces of the tube not only makes this assumption very improbable, but violates an essential condition upon which the theory of thin shells is based. When one surface only is subjected to pressure, there is reason to believe that BRYAN'S solution is substantially correct; but no treatment can be looked upon as rigorous which neglects the cross-stresses in the material.

The experimental researches of A. P. CARMAN|| and R. T. STEWART¶ have revived interest in this problem, since they offer the first information which has been obtained as to the behaviour under practical conditions of tubes which in circularity, uniformity of thickness and homogeneity are fair approximations to the ideal tube of theoretical analysis.**

We commence our discussion by considering the stability of a thin cylindrical tube, subjected to the combined action of end and surface pressures. We shall thus be able to derive the required solutions for the thin tubular strut, and for a boiler flue without end-thrust, as particular cases, and from the general solution we may obtain indications of the way in which end-thrust tends to promote the collapse of a boiler flue.

In the most general form of the boiler-flue problem, as enunciated by BASSET,†† pressures are acting on both surfaces of the tube, and we shall therefore investigate conditions for neutral stability in a tube subjected to the following system of stresses:—

- (i.) An end-thrust of total amount S , uniformly distributed;
- (ii.) An external hydrostatic pressure, of intensity p_1 ; and
- (iii.) An internal hydrostatic pressure, of intensity p_2 .

* 'Physikalische Zeitschrift,' XII. (1911), p. 241.

† 'Inst. Civ. Eng. Ireland,' 1910.

‡ 'Proc. Lond. Math. Soc.,' XXIV. (1893), p. 208.

§ 'Phil. Mag.,' XXXIV. (1892), p. 221.

|| "Resistance of Tubes to Collapse," 'Bulletin of the Univ. of Illinois,' No. 17, 1906.

¶ "Collapsing Pressures of Bessemer Steel Lap-Welded Tubes," 'Trans. American Soc. Mech. Eng.,' 1906, p. 730.

** The experiments of FAIRBAIRN were restricted to tubes which were constructed from sheet metal, with brazed and riveted seams.

†† *Loc. cit.*, p. 223.

We shall consider a tube of indefinite length, of which the inner and outer radii are

$$\alpha \pm t \quad (\text{so that the thickness is } 2t),$$

and we shall write

$$\tau \text{ for the ratio } \frac{t}{\alpha}.$$

The corresponding stress-system, for the position of equilibrium, is easily obtained.*

We have

$$\left. \begin{aligned} \widehat{rr} &= -\frac{1}{4\tau} \left[\mathfrak{P}_1 (1+\tau)^2 - \mathfrak{P}_2 (1-\tau)^2 - \frac{\alpha^2}{r^2} (\mathfrak{P}_1 - \mathfrak{P}_2) (1-\tau^2)^2 \right], \\ \widehat{\theta\theta} &= -\frac{1}{4\tau} \left[\mathfrak{P}_1 (1+\tau)^2 - \mathfrak{P}_2 (1-\tau)^2 + \frac{\alpha^2}{r^2} (\mathfrak{P}_1 - \mathfrak{P}_2) (1-\tau^2)^2 \right], \end{aligned} \right\} \dots \quad (51)$$

and

$$\widehat{zz} = -\frac{\mathfrak{S}}{4\pi\alpha t}.$$

It can also be shown that e_3 is constant, and equations (46–48) may therefore be taken to express the conditions of neutral equilibrium. The degree of approximation to which these equations have been obtained (p. 206) will be maintained for the rest of this paper, *i.e.*, terms of order $\frac{\widehat{rr}^2}{C^2} u' \dots$ will be neglected. They may also be written as follows:—

$$\begin{aligned} & 2 \frac{m-1}{m-2} \left(\frac{\partial^2 u'}{\partial r^2} + \frac{1}{r} \frac{\partial u'}{\partial r} - \frac{u'}{r^2} \right) + \left(1 + \frac{A}{2} \right) \frac{1}{r^2} \frac{\partial^2 u'}{\partial \theta^2} + \left\{ 1 + \frac{A}{4} \left(1 + \frac{\sigma \alpha^2}{r^2} \right) - \frac{B}{4} \right\} \frac{\partial^2 u'}{\partial z^2} \\ & + \left(\frac{m}{m-2} - \frac{A}{2} \right) \frac{1}{r} \frac{\partial^2 v'}{\partial r \partial \theta} - \left(\frac{3m-4}{m-2} + \frac{A}{2} \right) \frac{1}{r^2} \frac{\partial v'}{\partial \theta} \\ & + \left\{ \frac{m}{m-2} - \frac{A}{4} \left(1 + \frac{\sigma \alpha^2}{r^2} \right) + \frac{B}{4} \right\} \frac{\partial^2 w'}{\partial z \partial r} = 0, \quad \dots \quad (52) \end{aligned}$$

$$\begin{aligned} & \left(\frac{m}{m-2} - \frac{A}{2} \right) \frac{1}{r} \frac{\partial^2 u'}{\partial r \partial \theta} + \left(\frac{3m-4}{m-2} + \frac{A}{2} \right) \frac{1}{r^2} \frac{\partial u'}{\partial \theta} \\ & + \left(1 + \frac{A}{2} \right) \left(\frac{\partial^2 v'}{\partial r^2} + \frac{1}{r} \frac{\partial v'}{\partial r} - \frac{v'}{r^2} \right) + 2 \frac{m-1}{m-2} \cdot \frac{1}{r^2} \frac{\partial^2 v'}{\partial \theta^2} \\ & + \left\{ 1 + \frac{A}{4} \left(1 - \frac{\sigma \alpha^2}{r^2} \right) - \frac{B}{4} \right\} \frac{\partial^2 v'}{\partial z^2} + \left\{ \frac{m}{m-2} - \frac{A}{4} \left(1 - \frac{\sigma \alpha^2}{r^2} \right) + \frac{B}{4} \right\} \frac{1}{r} \frac{\partial^2 w'}{\partial \theta \partial z} = 0, \quad (53) \end{aligned}$$

and

$$\begin{aligned} & \left\{ \frac{m}{m-2} - \frac{A}{4} \left(1 + \frac{\sigma \alpha^2}{r^2} \right) + \frac{B}{4} \right\} \frac{\partial^2 u'}{\partial z \partial r} + \left\{ \frac{m}{m-2} - \frac{A}{4} \left(1 - \frac{\sigma \alpha^2}{r^2} \right) + \frac{B}{4} \right\} \frac{1}{r} \frac{\partial u'}{\partial z} \\ & + \left\{ \frac{m}{m-2} - \frac{A}{4} \left(1 - \frac{\sigma \alpha^2}{r^2} \right) + \frac{B}{4} \right\} \frac{1}{r} \frac{\partial^2 v'}{\partial \theta \partial z} + \left\{ 1 + \frac{A}{4} \left(1 + \frac{\sigma \alpha^2}{r^2} \right) - \frac{B}{4} \right\} \frac{\partial^2 w'}{\partial r^2} \\ & + \left\{ 1 + \frac{A}{4} \left(1 - \frac{\sigma \alpha^2}{r^2} \right) - \frac{B}{4} \right\} \frac{1}{r} \frac{\partial w'}{\partial r} + \left\{ 1 + \frac{A}{4} \left(1 - \frac{\sigma \alpha^2}{r^2} \right) - \frac{B}{4} \right\} \frac{1}{r^2} \frac{\partial^2 w'}{\partial \theta^2} + 2 \frac{m-1}{m-2} \cdot \frac{\partial^2 w'}{\partial z^2} = 0, \quad (54) \end{aligned}$$

* LOVE, 'Mathematical Theory of Elasticity' (2nd edition), § 100.

where

$$\left. \begin{aligned} A &= -\frac{1}{4C\tau} [\mathfrak{P}_1(1+\tau)^2 - \mathfrak{P}_2(1-\tau)^2], \\ B &= \frac{\mathfrak{S}}{4\pi a t C}, \\ \sigma &= -\frac{(\mathfrak{P}_1 - \mathfrak{P}_2)(1-\tau^2)^2}{\mathfrak{P}_1(1+\tau)^2 - \mathfrak{P}_2(1-\tau)^2}. \end{aligned} \right\} \dots \dots \dots (55)$$

and

We may assume a solution for equations (52-54) of the form*

$$\left. \begin{aligned} u' &= \Sigma \left[U_{k,q} \sin k(\theta + \theta_k) \sin \frac{q}{a}(z + z_q) \right] \\ v' &= \Sigma \left[V_{k,q} \cos k(\theta + \theta_k) \sin \frac{q}{a}(z + z_q) \right] \\ w' &= \Sigma \left[W_{k,q} \sin k(\theta + \theta_k) \cos \frac{q}{a}(z + z_q) \right] \end{aligned} \right\} \dots \dots \dots (56)$$

where k must be integral, and $U_{k,q}$, $V_{k,q}$, $W_{k,q}$ are functions of r only, which satisfy the differential equations

$$\begin{aligned} &\left[2 \frac{m-1}{m-2} \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} \right) - \frac{k^2}{r^2} \left(1 + \frac{A}{2} \right) - \frac{q^2}{a^2} \left\{ 1 + \frac{A}{4} \left(1 + \frac{\sigma a^2}{r^2} \right) - \frac{B}{4} \right\} \right] U_{k,q} \\ &\quad - k \left[\left(\frac{m}{m-2} - \frac{A}{2} \right) \frac{1}{r} \frac{d}{dr} - \left(\frac{3m-4}{m-2} + \frac{A}{2} \right) \frac{1}{r^2} \right] V_{k,q} \\ &\quad - \frac{q}{a} \left[\frac{m}{m-2} - \frac{A}{4} \left(1 + \frac{\sigma a^2}{r^2} \right) + \frac{B}{4} \right] \frac{d}{dr} W_{k,q} = 0, \dots \dots \dots (57) \end{aligned}$$

$$\begin{aligned} &k \left[\left(\frac{m}{m-2} - \frac{A}{2} \right) \frac{1}{r} \frac{d}{dr} + \left(\frac{3m-4}{m-2} + \frac{A}{2} \right) \frac{1}{r^2} \right] U_{k,q} \\ &\quad + \left[\left(1 + \frac{A}{2} \right) \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} \right) - 2 \frac{m-1}{m-2} \cdot \frac{k^2}{r^2} - \frac{q^2}{a^2} \left\{ 1 + \frac{A}{4} \left(1 - \frac{\sigma a^2}{r^2} \right) - \frac{B}{4} \right\} \right] V_{k,q} \\ &\quad - k \frac{q}{a} \left[\frac{m}{m-2} - \frac{A}{4} \left(1 - \frac{\sigma a^2}{r^2} \right) + \frac{B}{4} \right] \frac{1}{r} W_{k,q} = 0, \dots \dots \dots (58) \end{aligned}$$

and

$$\begin{aligned} &\frac{q}{a} \left[\left\{ \frac{m}{m-2} - \frac{A}{4} \left(1 + \frac{\sigma a^2}{r^2} \right) + \frac{B}{4} \right\} \frac{d}{dr} + \left\{ \frac{m}{m-2} - \frac{A}{4} \left(1 - \frac{\sigma a^2}{r^2} \right) + \frac{B}{4} \right\} \frac{1}{r} \right] U_{k,q} \\ &\quad k \frac{q}{a} \left[\frac{m}{m-2} - \frac{A}{4} \left(1 - \frac{\sigma a^2}{r^2} \right) + \frac{B}{4} \right] \frac{1}{r} V_{k,q} \\ &\quad + \left[\left\{ 1 + \frac{A}{4} \left(1 + \frac{\sigma a^2}{r^2} \right) - \frac{B}{4} \right\} \frac{d^2}{dr^2} + \left\{ 1 + \frac{A}{4} \left(1 - \frac{\sigma a^2}{r^2} \right) - \frac{B}{4} \right\} \left(\frac{1}{r} \frac{d}{dr} - \frac{k^2}{r^2} \right) \right. \\ &\quad \left. - 2 \frac{m-1}{m-2} \cdot \frac{q^2}{a^2} \right] W_{k,q} = 0. \dots \dots (59) \end{aligned}$$

It is easy to show that the phase-relations assumed in equations (56) are necessary.

The boundary conditions now require investigation. From the consideration that the cylindrical boundary surfaces of the tube must continue to be tangent to principal planes of stress, in any possible type of distortion, we deduce the conditions

$$\left. \begin{aligned} m_1 &= 0 \\ n_1 &= 0 \end{aligned} \right\}, \text{ identically, when } r = a \pm t. \quad \dots \quad (60)$$

The other boundary conditions are more complex. Since the pressures acting on the surfaces of the tube are hydrostatic, it is clear that *the radial stress, as defined on p. 193, is increased at points on the boundary surfaces of the tube where the distortion involves positive extension.* In the notation employed above, we have

$$\begin{aligned} \widehat{rr}' &= -\mathfrak{P}_1, \quad \text{when } r = a+t, \\ &= -\mathfrak{P}_2, \quad \text{when } r = a-t, \end{aligned}$$

and from (43) we deduce the following equations, *which must be satisfied identically*,*

$$\left. \begin{aligned} &\frac{2}{m-2} \left[(m-1) \frac{\partial u'}{\partial r} + \frac{u'}{r} + \frac{1}{r} \frac{\partial v'}{\partial \theta} + \frac{\partial w'}{\partial z} \right] \\ &= \left\{ \begin{aligned} &-\frac{\mathfrak{P}_1}{C} \left[\frac{u'}{r} + \frac{1}{r} \frac{\partial v'}{\partial \theta} + \frac{\partial w'}{\partial z} \right], \quad \text{when } r = a+t, \\ &-\frac{\mathfrak{P}_2}{C} \left[\frac{u'}{r} + \frac{1}{r} \frac{\partial v'}{\partial \theta} + \frac{\partial w'}{\partial z} \right], \quad \text{,, } r = a-t, \end{aligned} \right\} \dots \quad (61)$$

Substituting from (56) in the identities (60) and (61), we finally obtain, as the required boundary conditions in $U_{k,q}$, $V_{k,q}$, and $W_{k,q}$,

$$\left. \begin{aligned} &\frac{k}{r} U_{k,q} + \left\{ \left(1 - A\sigma \frac{\alpha^2}{r^2} \right) \frac{d}{dr} - \frac{1}{r} \right\} V_{k,q} = 0, \\ &\frac{q}{a} U_{k,q} + \left\{ 1 - \frac{A}{2} \left(1 + \sigma \frac{\alpha^2}{r^2} \right) - \frac{B}{2} \right\} \frac{d}{dr} W_{k,q} = 0, \end{aligned} \right\} \dots \quad (62)$$

and

$$2 \frac{m-1}{m-2} \frac{d}{dr} U_{k,q} + \left\{ \frac{2}{m-2} - A \left(1 + \frac{\sigma \alpha^2}{r^2} \right) \right\} \left(\frac{1}{r} U_{k,q} - \frac{k}{r} V_{k,q} - \frac{q}{a} W_{k,q} \right) = 0,$$

when $r = a \pm t$.

* In obtaining these equations it should be noticed that before distortion occurs $-\mathfrak{P}_1$ and $-\mathfrak{P}_2$ are the values at the boundary of

$$\frac{\widehat{rr}}{(1+\epsilon_2)(1+\epsilon_3)},$$

and not of \widehat{rr} , if we retain the significance for \widehat{rr} which was assumed on p. 193. The distinction is not really needed for the approximation of the following work, but it may lead to confusion if neglected.

The differential relations (57–59), with the boundary conditions (62), are theoretically sufficient for an exact solution of our problem: we shall, however, content ourselves with approximate solutions for $(\mathfrak{P}_1 - \mathfrak{P}_2)$ and \mathfrak{S} , correct to terms in τ^3 . To obtain these, we assume solutions for $U_{k,q}$, $V_{k,q}$, $W_{k,q}$, in series of ascending powers of the quantity $(r - \alpha)$. Thus we write

$$\left. \begin{aligned} U_{k,q} &= \xi_0 + \xi_1 \left(\frac{h}{\alpha}\right) + \frac{\xi_2}{2!} \left(\frac{h}{\alpha}\right)^2 + \dots \\ V_{k,q} &= \eta_0 + \eta_1 \left(\frac{h}{\alpha}\right) + \frac{\eta_2}{2!} \left(\frac{h}{\alpha}\right)^2 + \dots \\ W_{k,q} &= \zeta_0 + \zeta_1 \left(\frac{h}{\alpha}\right) + \frac{\zeta_2}{2!} \left(\frac{h}{\alpha}\right)^2 + \dots \end{aligned} \right\}, \dots \dots \dots (63)$$

where $r = \alpha + h$.

We may now derive, from equations (57–59), any required number of relations between the undetermined coefficients $\xi_0 \dots \eta_0 \dots \zeta_0 \dots$, and the boundary conditions (62) take the form of equations in series of ascending powers of the small quantity τ , in which the sums of the odd and of the even powers must vanish separately. If we neglect in these equations terms of order higher than some definite power of τ , we may obtain corresponding approximations to the values of A and B , by the elimination of the undetermined coefficients.

The approximate boundary conditions, correct to terms in τ^2 , are*

$$k\xi_0 + k\frac{\tau^2}{2}\xi_2 - \eta_0 + \{1 - A(\sigma - \tau^2)\}\eta_1 + \left(\frac{1}{2} - A\right)\tau^2\eta_2 + (1 + A)\frac{\tau^2}{2}\eta_3 = 0, \dots \dots \dots (64)$$

$$k\xi_1 + k\frac{\tau^2}{6}\xi_3 + A(\sigma - \tau^2)\eta_1 + \{1 - A(\sigma - \tau^2)\}\eta_2 + \left(\frac{1}{3} - \frac{A}{2}\right)\tau^2\eta_3 + (1 + A)\frac{\tau^2}{6}\eta_4 = 0, \dots \dots (65)$$

$$q\xi_0 + q\frac{\tau^2}{2}\xi_2 + \left\{1 - \frac{A(1 + \sigma) + B}{2} + \frac{3}{2}A\tau^2\right\}\xi_1 - A\tau^2\xi_2 + \left(1 - \frac{B}{2}\right)\frac{\tau^2}{2}\xi_3 = 0, \dots \dots \dots (66)$$

$$q\xi_1 + q\frac{\tau^2}{6}\xi_3 + A(\sigma - 2\tau^2)\xi_1 + \left\{1 - \frac{A(1 + \sigma) + B}{2} + \frac{3}{2}A\tau^2\right\}\xi_2 - \frac{1}{2}A\tau^2\xi_3 + \left(1 - \frac{B}{2}\right)\frac{\tau^2}{6}\xi_4 = 0, \dots \dots (67)$$

$$\begin{aligned} &\left\{1 - \frac{m-2}{2}A(1 + \sigma - 3\tau^2)\right\}\xi_0 + \{m-1 - (m-2)A\tau^2\}\xi_1 + (2m-1)\frac{\tau^2}{2}\xi_2 + (m-1)\frac{\tau^2}{2}\xi_3 \\ &\quad - k\left\{1 - \frac{m-2}{2}A(1 + \sigma - 3\tau^2)\right\}\eta_0 + k(m-2)A\tau^2\eta_1 - k\frac{\tau^2}{2}\eta_2 \\ &\quad - q\left\{1 - \frac{m-2}{2}A(1 + \sigma - \tau^2)\right\}\xi_0 - q\{1 - (m-2)A\}\tau^2\xi_1 - q\frac{\tau^2}{2}\xi_2 = 0, \dots \dots \dots (68) \end{aligned}$$

* In deriving these boundary conditions it is to be noticed that σ is to a first approximation equal to -1 , so that to our approximation $-\tau^2$ may be written for $\sigma\tau^2$.

and

$$\begin{aligned}
(m-2) A (\sigma - 2\tau^2) \xi_0 + \left\{ m - \frac{m-2}{2} A (1 + \sigma - 3\tau^2) \right\} \xi_1 + \left\{ m-1 - \frac{m-2}{2} A \tau^2 \right\} \xi_2 \\
+ (3m-2) \frac{\tau^2}{6} \xi_3 + (m-1) \frac{\tau^2}{6} \xi_4 - (m-2) k A (\sigma - 2\tau^2) \eta_0 - k \left\{ 1 - \frac{m-2}{2} A (1 + \sigma - 3\tau^2) \right\} \eta_1 \\
+ (m-2) k A \frac{\tau^2}{2} \eta_2 - k \frac{\tau^2}{6} \eta_3 - q \left\{ 1 - \frac{m-2}{2} A (1 - \sigma + \tau^2) \right\} \xi_0 \\
- q \left\{ 1 - \frac{m-2}{2} A (1 + \sigma - \tau^2) \right\} \xi_1 - q \left\{ 1 - (m-2) A \right\} \frac{\tau^2}{2} \xi_2 - q \frac{\tau^2}{6} \xi_3 = 0, \quad \dots \quad (69)
\end{aligned}$$

and these equations involve the fifteen coefficients

$$\xi_0 \dots \xi_4, \quad \eta_0 \dots \eta_4, \quad \xi_0 \dots \xi_4.$$

From equations (57-59) we may obtain nine other relations between these coefficients, as follows:—

$$\begin{aligned}
- \left[2 \frac{m-1}{m-2} + k^2 \left(1 + \frac{A}{2} \right) + q^2 \left(1 + \frac{A-B}{4} + \sigma \frac{A}{4} \right) \right] \xi_0 + 2 \frac{m-1}{m-2} \xi_1 + 2 \frac{m-1}{m-2} \xi_2 \\
+ k \left[\frac{3m-4}{m-2} + \frac{A}{2} \right] \eta_0 - k \left[\frac{m}{m-2} - \frac{A}{2} \right] \eta_1 \\
- q \left[\frac{m}{m-2} - \frac{A-B}{4} - \sigma \frac{A}{4} \right] \xi_1 = 0, \quad \dots \quad (70)
\end{aligned}$$

$$\begin{aligned}
2 \left[2 \frac{m-1}{m-2} + k^2 \left(1 + \frac{A}{2} \right) + q^2 \sigma \frac{A}{4} \right] \xi_0 - \left[4 \frac{m-1}{m-2} + k^2 \left(1 + \frac{A}{2} \right) + q^2 \left(1 + \frac{A-B}{4} + \sigma \frac{A}{4} \right) \right] \xi_1 \\
+ 2 \frac{m-1}{m-2} \xi_2 + 2 \frac{m-1}{m-2} \xi_3 - 2k \left[\frac{3m-4}{m-2} + \frac{A}{2} \right] \eta_0 + 4 \frac{m-1}{m-2} k \eta_1 - k \left[\frac{m}{m-2} - \frac{A}{2} \right] \eta_2 \\
- q \sigma \frac{A}{2} \xi_1 - q \left[\frac{m}{m-2} - \frac{A-B}{4} - \sigma \frac{A}{4} \right] \xi_2 = 0, \quad \dots \quad (71)
\end{aligned}$$

$$\begin{aligned}
-3 \left[2 \frac{m-1}{m-2} + k^2 \left(1 + \frac{A}{2} \right) + q^2 \sigma \frac{A}{4} \right] \xi_0 + 2 \left[3 \frac{m-1}{m-2} + k^2 \left(1 + \frac{A}{2} \right) + q^2 \sigma \frac{A}{4} \right] \xi_1 \\
- \frac{1}{2} \left[6 \frac{m-1}{m-2} + k^2 \left(1 + \frac{A}{2} \right) + q^2 \left(1 + \frac{A-B}{4} + \sigma \frac{A}{4} \right) \right] \xi_2 + \frac{m-1}{m-2} \xi_3 + \frac{m-1}{m-2} \xi_4 \\
+ 3k \left[\frac{3m-4}{m-2} + \frac{A}{2} \right] \eta_0 - k \left[\frac{7m-8}{m-2} + \frac{A}{2} \right] \eta_1 + \frac{1}{2} k \left[\frac{5m-4}{m-2} - \frac{A}{2} \right] \eta_2 \\
- \frac{1}{2} k \left[\frac{m}{m-2} - \frac{A}{2} \right] \eta_3 + \frac{3}{4} q \sigma A \xi_1 - q \sigma \frac{A}{2} \xi_2 - \frac{1}{2} q \left[\frac{m}{m-2} - \frac{A-B}{4} - \sigma \frac{A}{4} \right] \xi_3 = 0, \quad \dots \quad (72)
\end{aligned}$$

$$\begin{aligned}
& k \left[\frac{3m-4}{m-2} + \frac{A}{2} \right] \xi_0 + k \left[\frac{m}{m-2} - \frac{A}{2} \right] \xi_1 \\
& - \left[1 + \frac{A}{2} + 2 \frac{m-1}{m-2} k^2 + q^2 \left(1 + \frac{A-B}{4} - \sigma \frac{A}{4} \right) \right] \eta_0 + \left[1 + \frac{A}{2} \right] \eta_1 + \left[1 + \frac{A}{2} \right] \eta_2 \\
& - kq \left[\frac{m}{m-2} - \frac{A-B}{4} + \sigma \frac{A}{4} \right] \zeta_0 = 0, \quad \dots \dots \dots (73)
\end{aligned}$$

$$\begin{aligned}
& -k \left[\frac{3m-4}{m-2} + \frac{A}{2} \right] \xi_0 + k \left[\frac{3m-4}{m-2} + \frac{A}{2} \right] \xi_1 + k \left[\frac{m}{m-2} - \frac{A}{2} \right] \xi_2 \\
& + \left[1 + \frac{A}{2} + 2 \frac{m-1}{m-2} k^2 - q^2 \left(1 + \frac{A-B}{4} + \sigma \frac{A}{4} \right) \right] \eta_0 \\
& \quad - \left[1 + \frac{A}{2} + 2 \frac{m-1}{m-2} k^2 + q^2 \left(1 + \frac{A-B}{4} - \sigma \frac{A}{4} \right) \right] \eta_1 \\
& + 2 \left[1 + \frac{A}{2} \right] \eta_2 + \left[1 + \frac{A}{2} \right] \eta_3 + kq\sigma \frac{A}{2} \zeta_0 - kq \left[\frac{m}{m-2} - \frac{A-B}{4} + \sigma \frac{A}{4} \right] \zeta_1 = 0, \quad \dots (74)
\end{aligned}$$

$$\begin{aligned}
& k \left[\frac{3m-4}{m-2} + \frac{A}{2} \right] \xi_0 - k \left[\frac{3m-4}{m-2} + \frac{A}{2} \right] \xi_1 + \frac{1}{2} k \left[\frac{3m-4}{m-2} + \frac{A}{2} \right] \xi_2 + \frac{1}{2} k \left[\frac{m}{m-2} - \frac{A}{2} \right] \xi_3 \\
& - \left[1 + \frac{A}{2} + 2 \frac{m-1}{m-2} k^2 - \sigma \frac{A}{4} q^2 \right] \eta_0 + \left[1 + \frac{A}{2} + 2 \frac{m-1}{m-2} k^2 - q^2 \left(1 + \frac{A-B}{4} + \sigma \frac{A}{4} \right) \right] \eta_1 \\
& - \frac{1}{2} \left[1 + \frac{A}{2} + 2 \frac{m-1}{m-2} k^2 + q^2 \left(1 + \frac{A-B}{4} - \sigma \frac{A}{4} \right) \right] \eta_2 + \frac{3}{2} \left[1 + \frac{A}{2} \right] \eta_3 + \frac{1}{2} \left[1 + \frac{A}{2} \right] \eta_4 \\
& - \frac{3}{4} kq\sigma A \zeta_0 + \frac{1}{2} kq\sigma A \zeta_1 - \frac{1}{2} kq \left[\frac{m}{m-2} - \frac{A-B}{4} + \sigma \frac{A}{4} \right] \zeta_2 = 0, \quad \dots \dots \dots (75)
\end{aligned}$$

$$\begin{aligned}
& q \left[\frac{m}{m-2} - \frac{A-B}{4} + \sigma \frac{A}{4} \right] \xi_0 + q \left[\frac{m}{m-2} - \frac{A-B}{4} - \sigma \frac{A}{4} \right] \xi_1 \\
& - kq \left[\frac{m}{m-2} - \frac{A-B}{4} + \sigma \frac{A}{4} \right] \eta_0 \\
& - \left[k^2 \left(1 + \frac{A-B}{4} - \sigma \frac{A}{4} \right) + 2 \frac{m-1}{m-2} q^2 \right] \zeta_0 + \left[1 + \frac{A-B}{4} - \sigma \frac{A}{4} \right] \zeta_1 \\
& \quad + \left[1 + \frac{A-B}{4} + \sigma \frac{A}{4} \right] \zeta_2 = 0, \quad \dots (76)
\end{aligned}$$

$$\begin{aligned}
& -q\sigma \frac{A}{2} \xi_0 + 2q \left[\frac{m}{m-2} - \frac{A-B}{4} + \sigma \frac{A}{4} \right] \xi_1 + q \left[\frac{m}{m-2} - \frac{A-B}{4} - \sigma \frac{A}{4} \right] \xi_2 \\
& + kq\sigma \frac{A}{2} \eta_0 - kq \left[\frac{m}{m-2} - \frac{A-B}{4} + \sigma \frac{A}{4} \right] \eta_1 \\
& + \left[k^2 \left(1 + \frac{A-B}{4} - \frac{3}{4} \sigma A \right) - 2 \frac{m-1}{m-2} q^2 \right] \zeta_0 + \left[\sigma \frac{A}{2} - k^2 \left(1 + \frac{A-B}{4} - \sigma \frac{A}{4} \right) - 2 \frac{m-1}{m-2} q^2 \right] \zeta_1 \\
& + 2 \left[1 + \frac{A-B}{4} - \sigma \frac{A}{4} \right] \zeta_2 + \left[1 + \frac{A-B}{4} + \sigma \frac{A}{4} \right] \zeta_3 = 0, \quad \dots \dots \dots (77)
\end{aligned}$$

$$\begin{aligned}
& \frac{3}{4}q\sigma A\xi_0 - \frac{3}{4}q\sigma A\xi_1 + \frac{3}{2}q \left[\frac{m}{m-2} - \frac{A-B}{4} + \sigma \frac{A}{4} \right] \xi_2 + \frac{1}{2}q \left[\frac{m}{m-2} - \frac{A-B}{4} - \sigma \frac{A}{4} \right] \xi_3 \\
& - \frac{3}{4}kq\sigma A\eta_0 + \frac{1}{2}kq\sigma A\eta_1 - \frac{1}{2}kq \left[\frac{m}{m-2} - \frac{A-B}{4} + \sigma \frac{A}{4} \right] \eta_2 \\
& - k^2 \left[1 + \frac{A-B}{4} - \frac{3}{2}\sigma A \right] \xi_0 + \left[k^2 \left(1 + \frac{A-B}{4} - \frac{3}{4}\sigma A \right) - \frac{3}{4}\sigma A - 2 \frac{m-1}{m-2} q^2 \right] \xi_1 \\
& + \frac{1}{2} \left[\frac{3}{2}\sigma A - k^2 \left(1 + \frac{A-B}{4} - \sigma \frac{A}{4} \right) - 2 \frac{m-1}{m-2} q^2 \right] \xi_2 + \frac{3}{2} \left[1 + \frac{A-B}{4} - \sigma \frac{A}{4} \right] \xi_3 \\
& + \frac{1}{2} \left[1 + \frac{A-B}{4} + \sigma \frac{A}{4} \right] \xi_4 = 0. \dots \dots \dots (78)
\end{aligned}$$

We may now eliminate the coefficients from equations (64-78), and obtain a determinantal equation, of fifteen rows, which gives a relation between A, B and the dimensions of the tube. This relation is the condition for neutral equilibrium of the initial stress-system, and is clearly correct to terms in τ^2 ; but by further consideration of the terms involved we may show that the labour which would be required for its complete evaluation is unnecessary, and as the fifteen-row determinant may be written down directly from the above equations it will not be given here.

Solution for Boiler Flue without End Thrust.

We shall begin by deriving a sufficiently approximate expression for the difference of pressure required to produce collapse of a tube, when there is no resultant end thrust or tension; and in the first case we shall deal with a form of collapse possible only in the case of tubes of infinite length. That is to say, we make B and q equal to zero in the fifteen-row determinant, which may then be reduced to one of ten rows.

In the latter determinant we may treat A as a quantity of order τ^2 ; for if A be put equal to zero, and the determinant be expanded, the terms which are independent of τ vanish identically. Hence $-A$ may be written for σA , and $A\tau^2$ may be neglected.

The ten-row determinant, simplified by these and other obvious modifications, is given on pp. 218 and 219. Expanding it from the top row, with the neglect of terms of order higher than τ^2 , we obtain

$$\frac{4m}{m-2} \left(\frac{m-1}{m-2} \right) \left[\tau^2 \left\{ m(k^2-1) + m - \frac{m}{3}(k^2+2) \right\} + 2(m-1)A \right] = 0,$$

whence

$$-A = \frac{\tau^2}{3} \cdot \frac{m}{m-1} (k^2-1). \dots \dots \dots (79)$$

0,	0,	$\frac{\tau^2}{2}$,	0,	0,
0,	1,	0,	$\frac{\tau^2}{6}$,	0,
-1,	$m-1$,	$(2m-1)\frac{\tau^2}{2}$,	$(m-1)\frac{\tau^2}{2}$,	0,
$(m-2)A$,	m ,	$m-1$,	$(3m-2)\frac{\tau^2}{6}$,	$\frac{\tau^2}{6}$,
$2\frac{m-1}{m-2}$,	$2\frac{m-1}{m-2}$,	$2\frac{m-1}{m-2}$,	0,	0,
$-4\frac{m-1}{m-2}$,	$-\left[4\frac{m-1}{m-2} + k^2\left(1 + \frac{A}{2}\right)\right]$,	$2\frac{m-1}{m-2}$,	$2\frac{m-1}{m-2}$,	0,
$6\frac{m-1}{m-2}$,	$2\left[3\frac{m-1}{m-2} + k^2\left(1 + \frac{A}{2}\right)\right]$,	$-\frac{1}{2}\left[6\frac{m-1}{m-2} + k^2\left(1 + \frac{A}{2}\right)\right]$,	$\frac{m-1}{m-2}$,	$\frac{1}{m-2}$,
$-2\frac{m-1}{m-2}$,	$\frac{m}{m-2} - \frac{A}{2}$,	0,	0,	0,
$2\frac{m-1}{m-2}$,	$\frac{3m-4}{m-2} + \frac{A}{2}$,	$\frac{m}{m-2} - \frac{A}{2}$,	0,	0,
$-2\frac{m-1}{m-2}$,	$-\left(\frac{3m-4}{m-2} + \frac{A}{2}\right)$,	$\frac{1}{2}\left(\frac{3m-4}{m-2} + \frac{A}{2}\right)$,	$\frac{1}{2}\left(\frac{m}{m-2} - \frac{A}{2}\right)$,	0,
1,	0,	0,	-1,	
0,	1,	0,	0,	
1,	0,	0,	0,	
0,	1,	0,	0,	
$1 - \frac{m-2}{2}A(1+\sigma)$,	$m-1$,	0,	$-k^2\left[1 - \frac{m-2}{2}A(1+\sigma)\right]$,	
$(m-2)A\sigma$,	$m - \frac{m-2}{2}A(1+\sigma)$,	1,	$-k^2(m-2)A\sigma$,	
$-\left[2\frac{m-1}{m-2} + k^2\left(1 + \frac{A}{2}\right) + q^2\left\{1 + \frac{A}{4}(1+\sigma)\right\}\right]$,	$2\frac{m-1}{m-2}$,	$\frac{2}{m-2}$,	$k^2\left(\frac{3m-4}{m-2} + \frac{A}{2}\right)$,	
$\frac{3m-4}{m-2} + \frac{A}{2}$,	$\frac{m}{m-2} - \frac{A}{2}$,	0,	$-\left[1 + \frac{A}{2} + 2\frac{m-1}{m-2}k^2 + q^2\left\{1 + \frac{A}{4}(1-\sigma)\right\}\right]$,	
$\frac{m}{m-2} - \frac{A}{4}(1-\sigma)$,	$\frac{m}{m-2} - \frac{A}{4}(1+\sigma)$,	0,	$-k^2\left[\frac{m}{m-2} - \frac{A}{4}(1-\sigma)\right]$,	

$-1,$	$\frac{A}{k^2},$	$\frac{\tau^2}{2},$	$\frac{\tau^2}{2},$	$0,$
$0,$	$-\frac{A}{k^2},$	$1+A,$	$\frac{\tau^2}{3},$	$\frac{\tau^2}{3},$
$-k^2,$	$-1,$	$-k^2 \frac{\tau^2}{2},$	$0,$	$0,$
$k^2(m-2)A,$	$-1+(m-2)A,$	$0,$	$-k^2 \frac{\tau^2}{6},$	$0,$
$k^2 \left(\frac{3m-4}{m-2} + \frac{A}{2} \right),$	$2+A,$	$0,$	$0,$	$0,$
$-2k^2 \left(\frac{3m-4}{m-2} + \frac{A}{2} \right),$	$-(2+A),$	$-k^2 \left(\frac{m}{m-2} - \frac{A}{2} \right),$	$0,$	$0,$
$3k^2 \left(\frac{3m-4}{m-2} + \frac{A}{2} \right),$	$2+A,$	$\frac{1}{2}k^2 \left(\frac{5m-4}{m-2} - \frac{A}{2} \right),$	$-\frac{1}{2}k^2 \left(\frac{m}{m-2} - \frac{A}{2} \right),$	$0,$
$-\left(1 + \frac{A}{2} + 2 \frac{m-1}{m-2} k^2 \right),$	$-2 \frac{m-1}{m-2},$	$1 + \frac{A}{2},$	$0,$	$0,$
$1 + \frac{A}{2} + 2 \frac{m-1}{m-2} k^2,$	$0,$	$2 \left(1 + \frac{A}{2} \right),$	$1 + \frac{A}{2},$	$0,$
$-\left(1 + \frac{A}{2} + 2 \frac{m-1}{m-2} k^2 \right),$	$0,$	$-\frac{1}{2} \left(1 + \frac{A}{2} + 2 \frac{m-1}{m-2} k^2 \right),$	$\frac{3}{2} \left(1 + \frac{A}{2} \right),$	$1 + \frac{A}{2},$
$1-A\sigma,$	$0,$	$0,$	$0,$	$0,$
$A\sigma,$	$1-A\sigma,$	$0,$	$0,$	$0,$
$0,$	$0,$	$0,$	$1-\frac{1}{2}A(1+\sigma),$	$0,$
$0,$	$0,$	$0,$	$A\sigma,$	$1-\frac{1}{2}A(1+\sigma),$
$0,$	$0,$	$-q^2 \left[1 - \frac{m-2}{2} A(1+\sigma) \right],$	$0,$	$0,$
$k^2 \left[1 - \frac{m-2}{2} A(1+\sigma) \right],$	$0,$	$-q^2 \left[1 - \frac{m-2}{2} A(1-\sigma) \right],$	$-q^2 \left[1 - \frac{m-2}{2} A(1+\sigma) \right],$	$0,$
$-k^2 \left(\frac{m}{m-2} - \frac{A}{2} \right),$	$0,$	$0,$	$-q^2 \left[\frac{m}{m-2} - \frac{A}{4} (1+\sigma) \right],$	$0,$
$1 + \frac{A}{2},$	$1 + \frac{A}{2},$	$-q^2 \left[\frac{m}{m-2} - \frac{A}{4} (1-\sigma) \right],$	$0,$	$0,$
$0,$	$0,$	$-\left[k^2 \left\{ 1 + \frac{A}{4} (1-\sigma) \right\} + 2 \frac{m-1}{m-2} q^2 \right],$	$1 + \frac{A}{4} (1-\sigma),$	$1 + \frac{A}{4} (1+\sigma),$

But to a first approximation

$$A = -\frac{1}{4C\tau}(\mathfrak{P}_1 - \mathfrak{P}_2),$$

so that we have

$$\begin{aligned} \mathfrak{P}_1 - \mathfrak{P}_2 &= \frac{4}{3} \cdot \frac{m}{m-1} C\tau^3 (k^2 - 1) \\ &= \frac{2}{3} \cdot \frac{m^2}{m^2 - 1} E (k^2 - 1) \left(\frac{t}{a}\right)^3, \dots \dots \dots (80) \end{aligned}$$

which agrees with BRYAN'S result.*

To complete our discussion of this problem we must consider types of distortion in which the axial wave-length is finite, and thus obtain a theoretical estimate of the strength of short flues with fixed ends. A solution giving A correctly to terms in τ^2 may be derived from the complete fifteen-row determinant; but we may show that for practical purposes the labour which this evaluation would entail is quite unnecessary.

We find first of all that those terms in the expression for A which are independent of τ contain q^4 as a factor. Now $2AC$ being approximately equal to the mean hoop stress in the tube before collapse, it is clear that A must in all cases of practical importance be a very small quantity. It follows that in the expanded equation the terms in A are of primary importance, and A^2 and higher powers may be neglected; further, since q must also be small, that terms in q^6 and higher powers of q may be neglected in comparison with terms in q^4 , and that of the terms in τ^2 those which involve q are negligible in comparison with the terms already found.

In accordance with these principles we may derive the terms which are required to complete our solution from a nine-row determinant, obtained by omitting terms in τ^2 from the general determinant. This simplified determinant is given on pp. 218 and 219. Further, we may neglect A^2 in the expansion, and in the coefficient of A retain only those terms which do not involve q ; we thus obtain the equation

$$\sigma A = \frac{m+1}{m} \frac{q^4}{k^4(k^2-1)} \dots \dots \dots (81)$$

But, by equations (55),

$$\sigma A = \frac{(\mathfrak{P}_1 - \mathfrak{P}_2)}{4C\tau} (1 - \tau^2)^2,$$

and therefore, to the approximation of equation (81),

$$\mathfrak{P}_1 - \mathfrak{P}_2 = 4 \frac{m+1}{m} C \frac{q^4}{k^4(k^2-1)} \cdot \frac{t}{a} = 2E \frac{q^4}{k^4(k^2-1)} \frac{t}{a} \dots \dots \dots (82)$$

* Cf. footnote, p. 209.

Combining this result with (80) we have, as our final expression for the pressure-difference which can produce collapse of the flue,

$$p_1 - p_2 = 2E \frac{t}{a} \left[\frac{q^4}{k^4(k^2-1)} + \frac{1}{3} \frac{m^2}{m^2-1} (k^2-1) \frac{t^2}{a^2} \right] \dots \dots \dots (83)$$

In this equation t/a is the ratio of the thickness to the diameter of the tube, and k is the number of lobes in the distorted form of its cross-section. The quantity q is connected with the axial wave-length λ of the distortion by the relation

$$q\lambda = 2\pi a. \dots \dots \dots (84)$$

We may imagine a flue subjected at its ends to constraints which merely keep the ends circular, without imposing any other restrictions upon the type of distortion.* In this case the end conditions may be written in the form

$$\left. \begin{matrix} w' = 0 \\ v' = 0 \end{matrix} \right\} \text{ when } z = \begin{cases} 0 \\ l \end{cases}, \dots \dots \dots (85)$$

and from (56) it is clear that l , the length of the flue, is equal to $\pi a/q$.

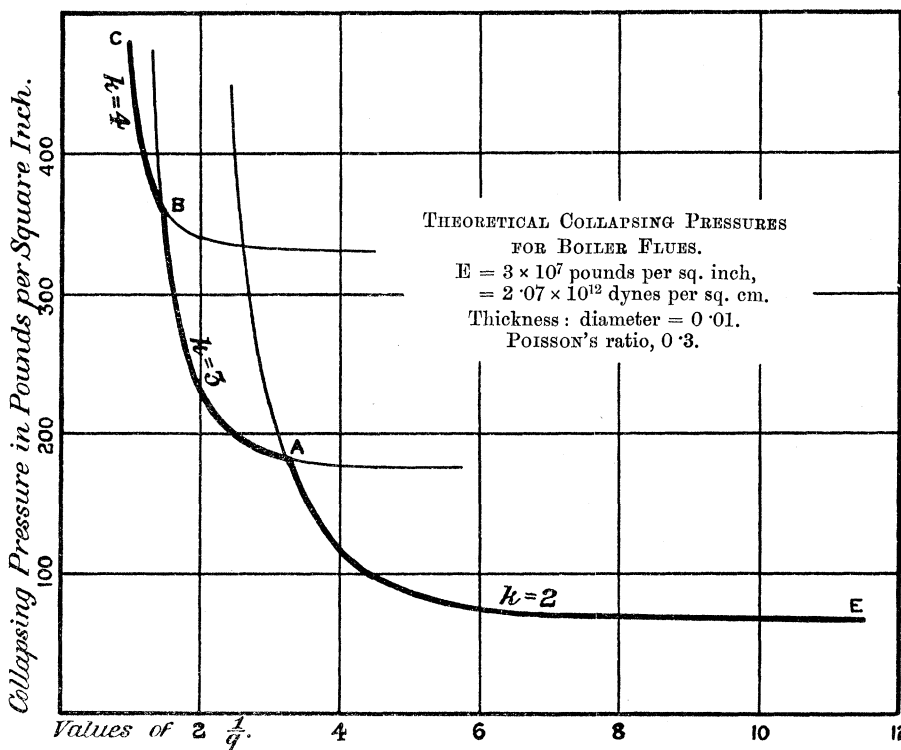


Fig. 3.

[* Added June 8.—Thin circular discs, inserted into the tube at its ends, but not fixed to it, would approximately realize these conditions.]

In practice, the end constraints will also tend to maintain the *cylindrical* form at the ends of the flue, and this effect will strengthen the tube, by an amount which is not easy to determine exactly. In any case we may say that

$$\frac{l}{a} \propto \frac{1}{q},$$

and we may illustrate the way in which the end effects die out by plotting the pressure differences ($\mathfrak{P}_1 - \mathfrak{P}_2$) against the quantity q^{-1} . To do this we must take some definite value of the ratio t/a , and plot different curves for the values 2, 3, ..., &c., of k . The result is shown by fig. 3, in which the following values have been assumed for the constants:—

$$\begin{aligned} E &= 3 \times 10^7 \text{ pounds per sq. inch,} \\ &= 2.07 \times 10^{12} \text{ dynes per sq. cm.} \end{aligned}$$

$$m = \frac{1}{3}, \quad \frac{t}{a} = \frac{1}{100}.$$

From an inspection of the different curves we see that long tubes will always tend to collapse into the two-lobed form, since the curve for $k = 2$ then gives the least value for the collapsing pressure, but that at a length corresponding to the point A the three-lobed distortion becomes natural to the tube, and for shorter lengths still, of which the point B gives the upper limit, the four-lobed form requires least pressure for its maintenance. Thus the true curve connecting pressure and length is the discontinuous curve CBAE, shown in the diagram by a thickened line.

Whatever be the relation between q and the length of the flue, it is clear that instability is theoretically possible in cases where the distortion involved is not even approximately “inextensional.” For if τ is sufficiently small, the collapsing pressure, as given by (83), need not involve elastic break-down in the position of equilibrium, even though the first (or “extensional”) term in (83) be equal to, or even greater than, the second. Of course, elastic break-down will occur by reason of the extension very soon after the commencement of the distortion. Nevertheless, failure in such a case must be regarded as due entirely to instability; for if this source of weakness were removed, effective resistance could be offered for an indefinite period to pressures which actually result in collapse.

Comparison with Experimental Results.

Although, as we have just remarked, it is theoretically possible for failure to occur by true elastic instability in comparatively short tubes, yet the relative dimensions of the tubes must be such as it would be quite impossible to test experimentally. In any practical case, instability will not occur until the properties of the material have been altered by overstrain, and the value of the pressure at collapse is therefore very much less than the foregoing theory would suggest.

It is, however, of interest to compare the general shape of the theoretical curve CBAE (fig. 3) with the results of experiment, and fig. 4 has been constructed for this purpose. It represents a number of tests conducted by the author upon seamless steel tube (0.028 inches thick and 1 inch in external diameter), and shows the relative amounts of resistance to external pressure offered by different lengths of tube. In these experiments (selected for fig. 4 from a more comprehensive series which is still

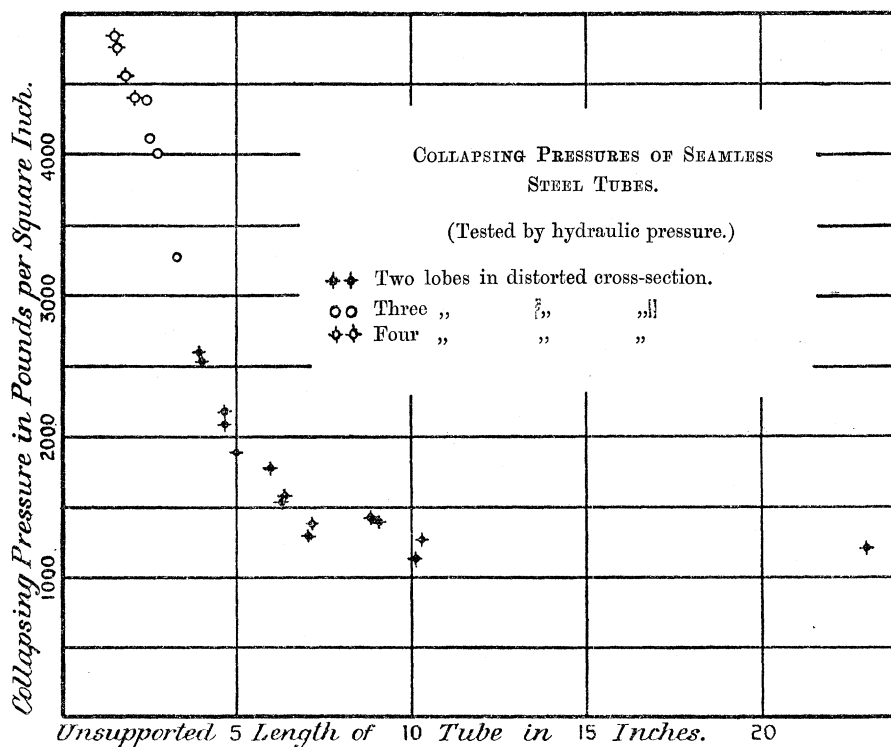


Fig. 4.

in progress) the ends of the tube were gripped by means of slightly conical plugs and sockets, the interior being kept in free connection with the atmosphere, and no attempt was made to balance the axial thrust due to hydrostatic pressure on the plugged end of the tube. Other experiments have shown that the existence of this thrust is not seriously important.

It will be seen that the general shape of the theoretical curve is well reproduced, as well as the changes in the number of lobes which characterize the distorted cross-section. Similar results to those of fig. 4 have been obtained by CARMAN,* but his experiments were not sufficiently numerous for a satisfactory comparison with the theoretical curve of fig. 3, his object in conducting them being merely to discover what is the limit of length beyond which the strength of a tube may be taken as

* Cf. footnote, p. 210.

sensibly the same for all lengths. The main interest both of CARMAN and of STEWART* was confined to tubes in excess of this limit, experiments on which may fairly be compared with the theoretical formula (80); their results showed that this formula gives a satisfactory estimate of the strength of very thin brass and steel tubes, but must not be taken as a basis for design throughout the whole range of dimensions employed in practice.

The experiments of FAIRBAIRN,† on the other hand, were restricted to tubes of such relatively small length that he failed to realize the existence of a definite minimum below which the strength of a tube, however long, will not fall. He also neglected the possibility of discontinuities in the curve of collapsing pressure at points where there is a change in the form of the distorted cross-section. In the light of these facts, figs. 3 and 4 help to explain his well-known formula, by which the collapsing pressure is given as inversely proportional to the length of the flue; for a curve of hyperbolic form will represent as well as any other single curve the scattered points of fig. 4, and trial shows that the hyperbola

$$\frac{p_1 - p_2}{q} = 464 \dots \dots \dots (86)$$

is very closely an envelope of the discontinuous curve CBAE in fig. 3, *in each case down to the point of least collapsing pressure.*

Validity of Investigation by the Theory of Thin Shells.

One important result of our investigation, which is apparently new, is shown by equation (83). It may be seen that collapse is practically dependent upon the pressure-difference alone, and that the absolute values of the pressures are immaterial. In view of this result, the objections raised by BASSET against BRYAN's treatment of the problem‡ require further consideration.

These objections are: first, that the ordinary expressions for the stress-couples in a plate or shell, in terms of the curvature of its middle surface, are not valid when the surfaces are subject to pressure; and secondly, that it is not legitimate to assume, as we must if sufficient equations are to be obtained, that the middle surface is unextended in a configuration of slight distortion. Hence the theory of thin shells is not applicable to this problem.

The above difficulties may be almost entirely overcome by a change in the method of investigation which is employed. It is customary to derive equations for the equilibrium of the distorted shell directly, and without reference to the position of equilibrium. Such procedure renders it necessary to make BRYAN's assumptions, that the middle surface is unextended, and that the usual expressions for the stress-couples

* Cf. footnote, p. 210.

† Cf. footnote, p. 209.

‡ See footnote, p. 210.

are valid. But we may also proceed, as in the foregoing discussion, by first determining the stress-system for the equilibrium position, and then deriving equations for an infinitesimal displacement. The stress-couples which appear in these equations will be due to the *additional stresses introduced by the distortion, and since these, to a first approximation at least, vanish at the surfaces of the tube, they will be given with sufficient accuracy by the usual expressions.* Moreover, when the distortion is two-dimensional (as in BASSET'S problem), the *change* in the "hoop" stress-resultant will be of an order which is negligible, so that the middle surface may be regarded as undergoing *no extension relatively to the equilibrium position*, even though its area may be sensibly changed in comparison with the unstrained configuration.*

The method of investigation just described, which follows the actual sequence of occurrences in the material, is suggested as in every way preferable to existing methods, for the investigation of any problem in elastic stability. For the present example, in particular, it leads to the same results as the more rigorous methods of this paper.

Comparison with Existing Formulae.

Previous discussion of the boiler-flue problem by analytical methods have, without exception, dealt with a tube subjected to pressure on one surface only, and almost all of them have been restricted to the case of an indefinitely long flue. Their results have, therefore, to be compared with our equation (80), when \mathfrak{P}_2 is zero. It will be found that this equation agrees with the formula obtained by BRYAN† and BASSET‡: FÖPPL'S formula† omits the factor $\frac{m^2}{m^2-1}$, which measures the increased resistance to flexure of a long tube as compared with a circular ring.

The more general formula may be compared with that of LORENZ,‡ if \mathfrak{P}_2 be put equal to zero. It will be found that there is a serious want of agreement in regard to both terms in the expression (83). In support of the latter result, it may be urged that LORENZ' solution gives for the indefinitely long flue a result which does not agree with equation (80) (and, as we have just noticed, this is supported by previous investigations), and which vanishes, not when $k = 1$, but when $k = 0$. Now the value 1, in the case of an infinitely long flue, corresponds to translation of the tube as a whole, without distortion, and the value 0 to a change in the diameter of the tube, without any departure from circularity. It is clear that the applied pressures can have no tendency to maintain such a form of distortion, so that LORENZ' formula can hardly be correct.

[* *Added June 8.*—The arguments of this section are more fully developed in a paper by the author "On the Collapse of Tubes by External Pressure," published in the 'Philosophical Magazine' for May, 1913 (pp. 687-698).]

† *Cf. footnote, p. 209.*

‡ *Cf. footnote, p. 210.*

The "Critical Length."

A. E. H. LOVE* has investigated the rate at which the strengthening effect of circular ends falls off when the length of a boiler flue is increased. His result suggests that at a distance which is great compared with the quantity \sqrt{at} the influence of the ends becomes negligible, and the flue collapses under sensibly the same pressure as a tube of infinite length; hence, in order that "collapse rings" may have any appreciable effect, their distance apart must not exceed some experimentally-determined multiple of this quantity.

The greatest length of tube over which the ends exert any appreciable strengthening influence, or the least length for which collapse is possible under a pressure sensibly equal to the critical pressure, has been called by Prof. LOVE† the "critical length." It is a conception of great importance in experimental work; for, as we have seen,‡ tests on any length of tube in excess of this limit may be taken to give the strength of an infinite length of the same tube, and their results compared with the theoretical formula (80)§: but as a basis for the spacing of "collapse rings" it is superseded by the theory of this paper, which yields an expression for the greatest length of tube consistent with stability, when the thickness and diameter of the flue, and also the collapsing pressure, are given; and Prof. LOVE has suggested to the author that it would be better now to employ the term "critical length" in this more general significance. As we have seen (p. 222), the length of the tube is some multiple of the quantity a/q , and we may therefore obtain from (83) the following formula:—

$$\text{Critical length} = \frac{M\alpha}{k \sqrt[4]{(k^2-1)} \left\{ \frac{\beta_1 - \beta_2}{2E} \frac{\alpha}{t} - \frac{1}{3} \frac{m^2}{m^2-1} (k^2-1) \frac{t^2}{a^2} \right\}}, \dots \quad (87)$$

where M is a constant, depending upon the type of the collapse ring, and k has that integral value which gives the least value for the right-hand expression of equation (87).

Before this subject is dismissed, it should be noticed that the theory of this paper does not support Prof. LOVE's estimate, mentioned above, of the rate of decay of end effects. The term in equation (83) which depends upon the length of the tube may be regarded as negligible, compared with the constant term, when the ratio

$$\frac{q^4}{k^4(k^2-1)} \bigg/ \left\{ \frac{1}{3} \frac{m^2}{m^2-1} (k^2-1) \frac{t^2}{a^2} \right\}$$

* 'Proc. Lond. Math. Soc.,' XXIV. (1893), p. 208.

† 'Theory of Elasticity' (2nd edition), § 337 (b).

‡ Page 224.

§ In this sense the term "critical length" has also been employed by CARMAN, who began his research by investigating the strengthening effects of the end plugs with which he sealed his tubes for test.

has some sufficiently small value; and $\frac{q}{\alpha}$ being inversely proportional to the length of the tube, we deduce for the "critical length," *in the original sense of the term*, an equation of the form

$$L = f \sqrt{\frac{\alpha^3}{t}},$$

where f is constant. Prof. LOVE, as has been said, has obtained an equation of the form

$$L = f' \sqrt{at},$$

which is very different; but he has informed the author that in the light of the above investigation (pp. 210–222) he does not regard his method as adequate.*

Solution for Tubular Strut: Special Case.

We may obtain another simplification of the general determinant to ten rows by taking a zero value for k . This corresponds to a type of distortion, possible in the case of a tubular strut, in which the axis remains straight and the cross-sections circular, the diameter varying in a sinusoidal manner.

The ten-row determinant for this case is given on pp. 228 and 229; the factor $-q^2$ has been cancelled from the sixth column, and terms in A have been omitted, so as to yield a result for tubes collapsed by end pressure alone. The expansion is only correct to terms of order τ^2 , and for a first approximation we may also neglect the square and higher powers of B, which must be small in any case of practical importance. Investigating first the terms which are independent of τ^2 , we obtain

$$B = 2 \frac{m+1}{m} \cdot \frac{1}{q^2} \cdot \dots \dots \dots (88)$$

We may now employ the substitution

$$B = 2 \frac{m+1}{m} \cdot \frac{1}{q^2} + B' \tau^2$$

in the determinant, and expand it from the top row, neglecting terms of higher order than τ^2 .

A considerable amount of unnecessary labour may be avoided by a preliminary examination of the relative importance of the various terms involved. It will be

[* *Added May 4.*—An argument in favour of the new formula may be drawn from physical considerations. The resistance offered by a tube to any given form of distortion is due partly to the extension and partly to the flexure which such distortion entails; and it is clear that the relative importance of the extensional part increases as the thickness is reduced. Hence, other things being equal, the effects of the ends, which necessitate extension of the middle surface, are more important in a thin than in a thick tube; that is to say, they are sensible over a greater length.]

1,	0,	$\frac{\tau^2}{2}$,	0,
0	1,	0,	$\frac{\tau^2}{6}$,
1,	$m-1$,	$(2m-1)\frac{\tau^2}{2}$,	$(m-1)\frac{\tau^2}{2}$,
0,	m ,	$m-1$,	$(3m-2)\frac{\tau^2}{6}$,
$-\left[2\frac{m-1}{m-2} + q^2\left(1 - \frac{B}{4}\right)\right]$,	$2\frac{m-1}{m-2}$,	$2\frac{m-1}{m-2}$,	0,
$4\frac{m-1}{m-2}$,	$-\left[4\frac{m-1}{m-2} + q^2\left(1 - \frac{B}{4}\right)\right]$,	$2\frac{m-1}{m-2}$,	$2\frac{m-1}{m-2}$,
$-6\frac{m-1}{m-2}$,	$6\frac{m-1}{m-2}$,	$-\frac{1}{2}\left[6\frac{m-1}{m-2} + q^2\left(1 - \frac{B}{4}\right)\right]$,	$\frac{m-1}{m-2}$,
$\frac{m}{m-2} + \frac{B}{4}$,	$\frac{m}{m-2} + \frac{B}{4}$,	0,	0,
0,	$2\left(\frac{m}{m-2} + \frac{B}{4}\right)$,	$\frac{m}{m-2} + \frac{B}{4}$,	0,
0,	0,	$\frac{3}{2}\left(\frac{m}{m-2} + \frac{B}{4}\right)$,	$\frac{1}{2}\left(\frac{m}{m-2} + \frac{B}{4}\right)$,

0,	0	$1 - \frac{B}{2},$	0,	$\left(1 - \frac{B}{2}\right) \frac{\tau^2}{2},$	0,
0,	0,	0,	$1 - \frac{B}{2},$	0,	$\left(1 - \frac{B}{2}\right) \frac{\tau^2}{3},$
0,	1,	$-q^2 \tau^2,$	$-q^2 \frac{\tau^2}{2},$	0,	0,
$\frac{\tau^2}{6},$	1,	$-q^2,$	$-q^2 \frac{\tau^2}{2},$	$-q^2 \frac{\tau^2}{6},$	0,
0,	0,	$-q^2 \left(\frac{m}{m-2} + \frac{B}{4}\right),$	0,	0,	0,
0,	0,	0,	$-q^2 \left(\frac{m}{m-2} + \frac{B}{4}\right),$	0,	0,
$\frac{1}{m-2},$	0,	0,	0,	$-\frac{1}{2} q^2 \left(\frac{m}{m-2} + \frac{B}{4}\right),$	0,
0,	$2 \frac{m-1}{m-2},$	$1 - \frac{B}{4},$	$1 - \frac{B}{4},$	0,	0,
0,	$2 \frac{m-1}{m-2},$	$-2 \frac{m-1}{m-2} q^2,$	$2 \left(1 - \frac{B}{4}\right),$	$1 - \frac{B}{4},$	0,
0,	0,	$-2 \frac{m-1}{m-2} q^2,$	$-\frac{m-1}{m-2} q^2,$	$\frac{3}{2} \left(1 - \frac{B}{4}\right),$	$1 - \frac{B}{4},$

found that B' contains terms in q^2 and $\frac{1}{q^2}$, as well as terms independent of q . Thus the complete expression for B is of the form

$$B = \frac{1}{q^2} \left(2 \frac{m+1}{m} + \alpha \tau^2 \right) + \tau^2 (\beta + \gamma q^2),$$

and it is clear that B has a minimum value when the axial wave-length has a finite value, given by

$$\gamma \tau^2 q^4 = 2 \frac{m+1}{m} + \alpha \tau^2.$$

This minimum value, which alone is of practical importance, is given, to a first approximation in terms of τ , by the equation

$$B_{\min.} = 2\tau \sqrt{2 \frac{m+1}{m} \gamma}, \quad \dots \dots \dots (89)$$

so that the determination of α and β is not required.

By expansion of the determinant we find

$$\gamma = \frac{2}{3} \frac{m}{m-1},$$

and from (55) we deduce, for the minimum thrust required to produce collapse,

$$\mathfrak{S}_{\min.} = 8\pi Et^2 \sqrt{\frac{1}{3} \cdot \frac{m^2}{m^2-1}} \dots \dots \dots (90)$$

This expression is correct to terms in t^2 .

Validity of Investigation by the Theory of Thin Shells.

A complete investigation of the tubular strut problem must deal with lobed forms of deformation, since it is possible that one of these may require a smaller end-pressure for its maintenance than the circular form treated above. We have, therefore, to obtain a general expression for B (when A is zero) in terms both of k and q .

The derivation of this expression, if we employ the rigorous methods of the present paper, will entail nothing less than the evaluation of the complete fifteen-row determinant; for the existence of a "favourite type of distortion," of finite axial wave-length, which we have noticed in the particular case ($k=0$), is found by practical experiment to be equally a feature of the lobed forms of distortion, and shows that the terms in τ^2 are important. Now it will be shown that the value of $\mathfrak{S}_{\min.}$, when $k=0$, may be obtained, correctly to terms in t^2 , by the ordinary theory of thin shells; and as there is no reason to believe that the latter theory will lead to

less accurate results when k has a finite value, it does not seem necessary to employ our more rigorous method, with the very laborious calculations which it entails. We shall therefore rely upon the approximate theory for the treatment of the tubular strut problem in its general form. Slight modifications in method will be introduced, as suggested above (pp. 224–225), and only the more important steps will be given here.

Solution by the Theory of Thin Shells: General Case.

We consider the stability of an element of the tube, originally bounded by the planes

$$\theta, \quad \theta + \delta\theta, \quad \text{and} \quad z, \quad z + \delta z,$$

as shown below—

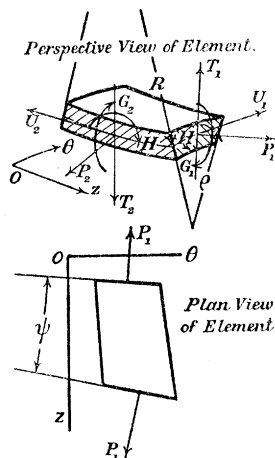


Fig. 5.

The other dimension of the element is the full thickness of the tube, denoted in this paper by $2t$. The radius of the middle surface is α .

The initial stress system is

$$P_1 = \text{const.} = -\frac{\mathfrak{S}}{2\pi\alpha} = [P_1] \text{ (say).}$$

In the distorted position this system produces a radial force on the element, of amount

$$\frac{1}{R} [P_1] \alpha \delta\theta \delta z,$$

where R is the radius of curvature of a section of the distorted element by an axial plane (see fig. 5).

It also produces a tangential force, in the direction of θ increasing, of amount

$$-[P_1] \alpha \psi \delta\theta,$$

where ψ (see fig. 5) = $\frac{1}{\alpha} \frac{\partial}{\partial \theta} (1 + e_{zz}) \delta z$.

The above system of distorting forces must be exactly balanced by the restoring system shown in the upper part of the figure. Hence we obtain the following equations of neutral stability :—

$$\left. \begin{aligned} \frac{[P_1]}{R} + \frac{\partial T_1}{\partial z} + \frac{1}{a} \frac{\partial T_2}{\partial \theta} - \frac{P_2}{a} &= 0, \\ -\frac{[P_1]}{a} \frac{\partial e_{zz}}{\partial \theta} + \frac{T_2}{a} + \frac{1}{a} \frac{\partial P_2}{\partial \theta} + \frac{\partial U_1}{\partial z} &= 0, \\ \frac{\partial P_1}{\partial z} + \frac{1}{a} \frac{\partial U_2}{\partial \theta} &= 0, \\ \frac{\partial G_1}{\partial z} - T_1 - \frac{1}{a} \frac{\partial H}{\partial \theta} &= 0, \\ \frac{1}{a} \frac{\partial G_2}{\partial \theta} + T_2 + \frac{\partial H}{\partial z} &= 0, \end{aligned} \right\} \dots \dots \dots (91)$$

Now R and e_{zz} may be expressed in terms of the displacements of the middle surface, as follows :—

$$\frac{1}{R} = \frac{\partial^2 u'}{\partial z^2}, \quad e_{zz} = \frac{\partial w'}{\partial z}; \quad \dots \dots \dots (92)$$

and the restoring system of stress resultants may also be expressed in terms of this system, as follows* :—

$$\left. \begin{aligned} P_1 &= \frac{3D}{t^2} \left[\frac{\partial w'}{\partial z} + \frac{1}{m\alpha} \left(u' + \frac{\partial v'}{\partial \theta} \right) \right], \\ P_2 &= \frac{3D}{t^2} \left[\frac{1}{a} \left(u' + \frac{\partial v'}{\partial \theta} \right) + \frac{1}{m} \frac{\partial w'}{\partial z} \right], \\ U_1 = U_2 &= \frac{3}{2} \frac{m-1}{m} \frac{D}{t^2} \left(\frac{\partial v'}{\partial z} + \frac{1}{a} \frac{\partial w'}{\partial \theta} \right), \\ G_1 &= -D \left[\frac{\partial^2 u'}{\partial z^2} + \frac{1}{m\alpha^2} \left(u' + \frac{\partial^2 u'}{\partial \theta^2} \right) \right], \\ G_2 &= D \left[\frac{1}{a^2} \left(u' + \frac{\partial^2 u'}{\partial \theta^2} \right) + \frac{1}{m} \frac{\partial^2 u'}{\partial z^2} \right], \\ H &= \frac{m-1}{m} D \frac{1}{a} \left(\frac{\partial^2 u'}{\partial \theta \partial z} - \frac{\partial v'}{\partial z} \right), \end{aligned} \right\} \dots \dots \dots (93)$$

where D is the quantity

$$\frac{2}{3} \frac{m^2}{m^2-1} Et^3 \dots \dots \dots (94)$$

* Cf. LOVE, 'Mathematical Theory of Elasticity' (second edition), Chap. XXIV. u' , v' , w' have the same significance as in the earlier part of this paper, except that they now refer to the middle surface, and are functions only of θ and z .

Eliminating T_1 and T_2 from equations (91), and substituting from (92) and (93), we have

$$\left. \begin{aligned} & \frac{u'}{\alpha^2} - \Psi \frac{\partial^2 u'}{\partial z^2} + \frac{1}{\alpha^2} \frac{\partial v'}{\partial \theta} + \frac{1}{ma} \frac{\partial w'}{\partial z} \\ & + \frac{t^2}{3} \left[\frac{1}{\alpha^4} \frac{\partial^2 u'}{\partial \theta^2} + \frac{1}{\alpha^4} \frac{\partial^4 u'}{\partial \theta^4} + \frac{2}{\alpha^2} \frac{\partial^2 u'}{\partial \theta^2 \partial z^2} + \frac{\partial^4 u'}{\partial z^4} + \frac{1}{m\alpha^2} \frac{\partial^2 u'}{\partial z^2} - 2 \frac{m-1}{m} \frac{1}{\alpha^2} \frac{\partial^3 v'}{\partial \theta \partial z^2} \right] = 0, \\ & \frac{1}{\alpha^2} \frac{\partial u'}{\partial \theta} + \frac{1}{\alpha^2} \frac{\partial^2 v'}{\partial \theta^2} + \frac{m-1}{2m} \frac{\partial^2 v'}{\partial z^2} + \left(\frac{m+1}{2m} - \Psi \right) \frac{1}{\alpha} \frac{\partial^2 w'}{\partial \theta \partial z} \\ & - \frac{t^2}{3} \left[\frac{1}{\alpha^4} \frac{\partial u'}{\partial \theta} + \frac{1}{\alpha^4} \frac{\partial^3 u'}{\partial \theta^3} + \frac{1}{\alpha^2} \frac{\partial^3 u'}{\partial \theta \partial z^2} - \frac{m-1}{m} \frac{1}{\alpha^2} \frac{\partial^2 v'}{\partial z^2} \right] = 0, \\ & \frac{1}{ma} \frac{\partial u'}{\partial z} + \frac{m+1}{2m} \frac{1}{\alpha} \frac{\partial^2 v'}{\partial \theta \partial z} + \frac{m-1}{2m} \frac{1}{\alpha^2} \frac{\partial^2 w'}{\partial \theta^2} + \frac{\partial^2 w'}{\partial z^2} = 0, \end{aligned} \right\} \quad (95)$$

where

$$\Psi = \frac{[P_1] t^2}{3D} = - \frac{m^2 - 1}{m^2} \frac{\mathfrak{S}}{4\pi a t E} \dots \dots \dots (96)$$

Assuming a solution of the type (56), we find, as the criterion for neutral stability,

$$\left| \begin{array}{ccc} 1 + \Psi q^2 + \frac{1}{3} \frac{t^2}{\alpha^2} \left[(k^2 + q^2)^2 - k^2 - \frac{1}{m} q^2 \right], & 1 + \frac{2}{3} \frac{m-1}{m} q^2 \frac{t^2}{\alpha^2}, & \frac{1}{m}, \\ k^2 + \frac{1}{3} k^2 \frac{t^2}{\alpha^2} [k^2 + q^2 - 1], & k^2 + \frac{m-1}{2m} q^2 + \frac{1}{3} \frac{m-1}{m} q^2 \frac{t^2}{\alpha^2}, & k^2 \left(\frac{m+1}{2m} - \Psi \right), \\ \frac{1}{m} q^2, & \frac{m+1}{2m} q^2; & \frac{m-1}{2m} k^2 + q^2, \end{array} \right| = 0 \dots \dots \dots (97)$$

This equation, in its expanded form, is

$$\begin{aligned} & \Psi^2 \left[\frac{m+1}{2m} k^2 q^4 \right] + \Psi \left[\frac{m-1}{2m} q^2 \{ (k^2 + q^2)^2 + k^2 \} \right] + \left(1 - \frac{1}{m^2} \right) \frac{m-1}{2m} q^4 \\ & + \frac{1}{3} \frac{t^2}{\alpha^2} \left\{ \Psi q^2 \left[\frac{m+1}{2m} k^2 \{ (k^2 + q^2)^2 - k^2 \} + \frac{m^2 - 7m + 4}{2m^2} k^2 q^2 + \frac{m-1}{m} q^4 \right] \right. \\ & + \frac{m-1}{2m} \left[(k^2 + q^2)^4 - \frac{q^6}{m} - k^2 \left\{ 2k^4 + 7k^2 q^2 + \frac{7m^2 + m - 2}{m^2} q^4 \right\} \right. \\ & \left. \left. + k^4 + 3k^2 q^2 + 2 \frac{m^2 - 1}{m^2} q^4 \right] \right\} = 0 \dots \dots \dots (98) \end{aligned}$$

Taking first the terms which are independent of t^2 , and neglecting the square of Ψ (which must be small), we find, as the first term in our solution,

$$-\Psi = \frac{m^2 - 1}{m^2} \frac{q^2}{(k^2 + q^2)^2 + k^2} \dots \dots \dots (99)$$

When $k = 0$, this becomes

$$-\Psi = \frac{m^2-1}{m^2} \cdot \frac{1}{q^2}, \quad \dots \dots \dots (100)$$

which shows that q must be great, if Ψ has a value possible in practice. Similarly, when $k > 0$, we see that q must be small, and the approximate expression in this case is

$$-\Psi = \frac{m^2-1}{m^2} \frac{q^2}{k^2(k^2+1)} \cdot \dots \dots \dots (101)$$

We may now determine sufficiently approximate expressions for the terms in t^2/α^2 , by treating q as great when $k = 0$, and as small when $k > 1$. That is to say, we retain only the highest and the lowest powers of q in the two cases.*

Thus, when $k = 0$, the important terms are

$$q^2\Psi + \left[\frac{m^2-1}{m^2} + \frac{1}{3}q^4 \frac{t^2}{\alpha^2} \right] = 0,$$

and we have

$$-\Psi = \frac{m^2-1}{m^2} \cdot \frac{1}{q^2} + \frac{1}{3}q^2 \frac{t^2}{\alpha^2},$$

or

$$\mathfrak{S} = 4\pi atE \left[\frac{1}{q^2} + \frac{1}{3} \frac{m^2}{m^2-1} q^2 \frac{t^2}{\alpha^2} \right] \cdot \dots \dots \dots (102)$$

When $k > 1$, the important terms are

$$q^2\Psi \left[k^2(k^2+1) + \frac{1}{3} \frac{t^2}{\alpha^2} \frac{m+1}{m-1} k^4(k^2-1) \right] \\ + \frac{m^2-1}{m^2} q^4 + \frac{1}{3} \frac{t^2}{\alpha^2} k^4(k^2-1)^2 = 0, \quad \dots \dots \dots (103)$$

whence, to terms in t^2/α^2 ,

$$-\Psi = \frac{m^2-1}{m^2} \frac{q^2}{k^2(k^2+1)} \left[1 + \frac{1}{3} \frac{t^2}{\alpha^2} \left\{ \frac{m^2}{m^2-1} \frac{k^4(k^2-1)^2}{q^4} - \frac{m+1}{m-1} k^2 \frac{k^2-1}{k^2+1} \right\} \right], \\ = \frac{m^2-1}{m^2} \frac{q^2}{k^2(k^2+1)} + \frac{1}{3} \frac{k^2(k^2-1)^2 t^2}{q^2 k^2+1 \alpha^2},$$

with sufficient accuracy, when q is small.

This leads to the result

$$\mathfrak{S} = \frac{4\pi atE}{k^2(k^2+1)} \left[q^2 + \frac{1}{3} \frac{m^2}{m^2-1} \frac{k^4(k^2-1)^2 t^2}{q^2 \alpha^2} \right] \cdot \dots \dots \dots (104)$$

For practical purposes only the stationary values of \mathfrak{S} are important. It is readily seen that the minimum value obtained from (102) agrees with (90), and is therefore

* In every case it is legitimate for practical purposes to neglect the term in Ψ^2 .

accurate as far as terms in t^2 ; we shall assume that (104) gives the same approximation, which for practical purposes is quite sufficient. We then find, for values of k other than 0 and 1, the expression

$$\mathfrak{S}_{\min.} = 8\pi Et^2 \sqrt{\frac{1}{3} \frac{m^2}{m^2-1} \frac{(k^2-1)}{(k^2+1)}} \dots \dots \dots (105)$$

When $k = 1$, the axis does not remain straight after distortion of the tube has occurred. This is the type of distortion (sometimes called "primary flexure") which was discussed by EULER, and it is easy to see that his result is identical with that of equation (104), which becomes in this case

$$\mathfrak{S}_{k=1} = 2\pi atEg^2 \dots \dots \dots (106)$$

The exact expression for the length of the tube, in terms of q , is not a matter of great importance in the present problem, because the wave-length corresponding to a minimum value of the collapsing pressure is in all cases small, and the strength of any strut of ordinary dimensions will therefore be given by equations (90) or (105), into which the length does not enter. As in the case of the boiler-flue problem, we

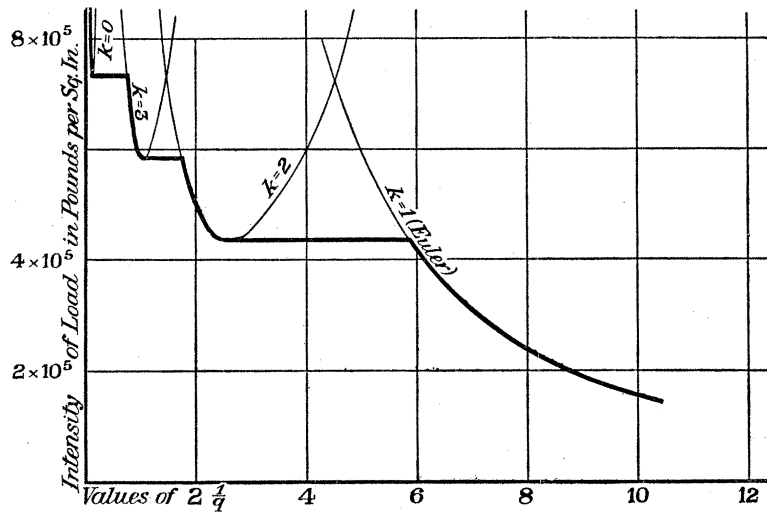


Fig. 6. Strength of Tubular Struts.

may illustrate the effects of length upon the collapsing thrust by plotting the intensity of stress, or $\mathfrak{S}/4\pi at$, against q^{-1} . For this purpose we must take some definite value of the ratio t/a , and draw separate curves for different integral values of k . The result is shown by fig. 6, in which the following values are assumed:—

$$\frac{t}{a} = \frac{1}{50}, \quad m = \frac{1}{3}.$$

$$E = 3 \times 10^7 \text{ pounds per sq. inch.} \\ = 2.07 \times 10^{12} \text{ dynes per sq. cm.}$$

From an inspection of these curves it is easily seen that as the axial wave-length increases the type of distortion which involves the least value for the collapsing thrust (and which the tube therefore tends naturally to assume) changes. For very short lengths we shall expect the circular type ($k = 0$); then, as the length increases, lobed forms of distortion, in which the value of k becomes less as the length increases. The limit is reached when $k = 1$; hence, the tendency of very long tubes is always to collapse in the manner discussed by EULER.

It is also to be noticed that those parts of the different curves which lie to the right of their lowest points have no practical significance. The actual curve, which shows the effect of length upon the value of the collapsing thrust, will approximate to the form shown in thick lines, since the wave-length (which varies as q^{-1}) will naturally not increase beyond that value which involves the least collapsing thrust.

Comparison with Existing Formulae.

The formulæ of equations (90) and (104) may be compared with the results of other discussions of this problem. Equation (90) has been obtained by LORENZ,* and LILLY* has given the same result, except that the factor $\sqrt{\frac{m^2}{m^2-1}}$ is omitted.†

The only existing solution for lobed forms of distortion is due to LORENZ,* and this is not in agreement with equation (104). In support of the latter formula it may be urged that LORENZ' formula does not agree with EULER'S result when $k = 1$.

It may also be remarked that the foregoing results for the tubular strut problem contradict BRYAN'S theorem, that a closed shell cannot fail by instability, because distortion would involve extension of the middle surface; for although the first terms in equations (102) and (104) are due solely to extension of the middle surface, yet the compressive stress at collapse, as given by (90) or (105), may be insufficient to produce elastic breakdown in the position of equilibrium, if the ratio $\frac{t}{a}$ has a sufficiently low value.

Stability of Tubes under Combined End and Surface Pressure.

We shall not treat this case in any detail, but it requires notice in connection with the "localization of collapse" which is observed in experiments conducted upon long tubes tested under hydrostatic pressure, the permanent distortion being generally confined to a portion only of the length of the tube. This result is not predicted by the theoretical formula (83), which suggests a steady fall in the value of the collapsing pressure as the wave-length increases; and a partial explanation may possibly be found in the fact that the method of test has generally left a wholly or partially

* Cf. footnote, p. 209.

† For a similar omission in a solution of the boiler-flue problem cf. p. 225.

unbalanced end-thrust, due to the water pressure acting upon the closed ends of the tube.

It is clear that the expansion of the general fifteen-row determinant will give an equation of the form

$$\alpha + \beta A + \gamma B + \delta A^2 + \epsilon AB + \zeta B^2 + \dots = 0,$$

where $\alpha, \beta, \gamma, \dots$ depend upon the dimensions of the tube and the type of the distortion. But in any practical case, as we have already observed, A and B must be very small quantities. It follows that an approximate solution may be obtained from the terms

$$\alpha + \beta A + \gamma B = 0. \quad \dots \dots \dots (107)$$

Let \mathfrak{P}_c and \mathfrak{S}_c be the values of the external pressure and of the end-thrust, each of which, acting alone, could produce collapse into the assumed type of distortion. Then equation (107) may clearly be written as follows:—

$$\mathfrak{P}_1 = \left(1 - \frac{\mathfrak{S}}{\mathfrak{S}_c}\right) \mathfrak{P}_c, \quad \dots \dots \dots (108)$$

where \mathfrak{P}_1 and \mathfrak{S} are the values of the external pressure and end-thrust which can produce collapse when acting *in conjunction*.

It may be seen from this equation that \mathfrak{P}_1 can have a minimum value for some finite value of the axial wave-length when, and only when, \mathfrak{S} exists. If the end-thrust be entirely unbalanced, we have

$$\mathfrak{S} = \pi a^2 (1 + \tau)^2 \mathfrak{P}_1, \quad \dots \dots \dots (109)$$

and the collapsing pressure may, in this case, be determined from equation (108).

GENERAL THEORY OF INSTABILITY IN MATERIALS OF FINITE STRENGTH.

The Practical Value of a Theory of Instability.

In the concluding section of this paper an attempt will be made to estimate the practical value of a theory of elastic instability; to suggest ways in which we may hope to increase this value; and to indicate the questions to which answers must be found in order that further advance may be possible.

The first point which must be noticed is the non-realization in practice of our conception of a "critical loading," owing to imperfections which always exist, and which violate our ideal assumptions. In any actual example the displacement of the system increases continuously with the load, and the system collapses at a smaller value of the load than our theory would dictate. It is necessary to inquire whether serious discrepancies are to be expected.

In some mechanical problems the effects of imperfections may be calculated. We may take, as an example, the system illustrated in fig. 1, and consider any one of the

many imperfections which occur in practice. For simplicity, let us assume that the sphere, bowl, and plunger are still smooth, rigid, and accurately formed, but that the line of thrust of the plunger is eccentric by an amount δ . It is easy to see from fig. 7 that the displacement of the sphere from the line of thrust of the plunger, when the system is in equilibrium under a load P , is

$$\left. \begin{aligned} d &= r \sin \theta = \delta + (R-r) \sin \phi, \\ \frac{P}{W} &= \frac{\tan \phi}{\tan \theta - \tan \phi}; \end{aligned} \right\} \dots \dots \dots (110)$$

and these equations enable us to trace the steady increase of the sphere's displacement as the load on the plunger is increased from a zero value.

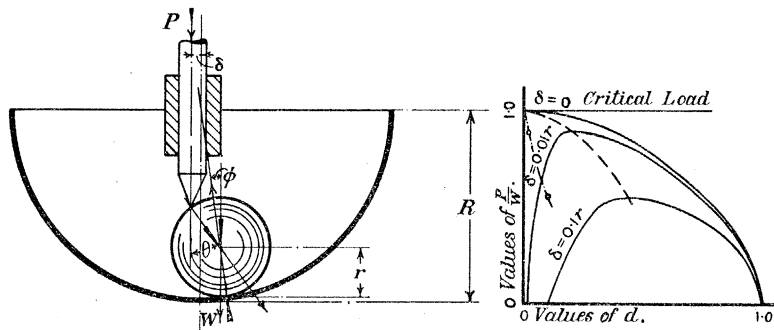


Fig. 7.

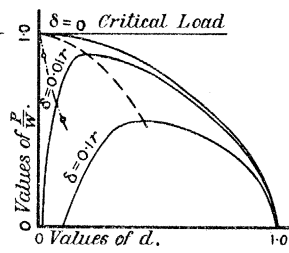


Fig. 8.

Thus in fig. 8 curves are drawn to connect P and d , for a value 3 of the ratio R/r , when the initial displacement δ has the values 0, $0.01r$, and $0.1r$ respectively. At the points on these curves for which P has a maximum value, "collapse" will occur, since the equilibrium then becomes unstable. The locus of these points is shown in the figure by a broken line, and a dot-and-dash line shows the connection between δ and the maximum value of P . From the latter curve it is evident that a small initial inaccuracy may cause a material reduction in the "collapsing load"; nevertheless the "critical load" gives a limit which will be more and more nearly attained as our experimental accuracy is improved, and its investigation is by no means useless for practical purposes.

When the problem is one of elastic stability, the discussion of imperfections by analytical methods will, in general, be beyond our power; but it is clear that similar remarks will apply. An "exchange of stabilities" at some "point of bifurcation"* must be regarded as a purely ideal conception, and in practice there will always be a steady increase of distortion as the load is increased, owing principally to practical imperfections of form. A strut, for example, may be very accurately loaded, if suitable methods are employed, but its centre-line will never be quite straight; the initial deflection which characterizes it may be regarded as composed of a series of

* H. POINCARÉ, 'Acta Mathematica,' 7. (1885), p. 259.

harmonic terms, and when the load is applied one of these harmonics will be developed very much more than the others, just as one constituent harmonic may be developed by "resonance" in an alternating current wave of irregular shape. In the ordinary strut problem this magnified harmonic is such that one-half wave occupies the length of the strut, but in other problems, such as that of the tubular strut, though there is always a "favourite" or "natural harmonic" which is especially magnified, its relation to the dimensions may be more complicated.* In any case the effects of practical imperfections of form might be studied, if the analytical difficulties could be surmounted, by investigating the rate at which the amplitude of this "natural harmonic" increases with the load, when its value in the initial configuration is given; and the results of the investigation might be shown graphically by curves of distortion, similar in character to the curves of fig. 8, in which the abscissæ represented the amplitude of the natural harmonic, and the ordinates represented the magnitude of the applied stress-system, or "load."

These "curves of distortion" are of considerable utility for the study of problems in elastic stability, even though their true form can only be guessed. They help us, for example, to explain, and in some degree to remedy, the serious discrepancy existing between EULER'S theory and the results of experiments on short struts. The discrepancy has often been attributed to practical imperfections of form; but it should hardly be necessary to point out that practical imperfections are likely to diminish rather than to increase in importance, as the dimensions of an elastic solid become more nearly comparable, so that they will never be more effective as causes of weakness than in struts of great length, which, as a matter of fact, give results in close agreement with EULER'S formula.

A more satisfactory explanation of this, and of similar discrepancies in other problems, may be found in the fact that the ordinary theory of elastic stability neglects the possibility of elastic break-down. If we attempt to draw "curves of distortion" for any single problem, we shall find that, apart from the other data of the problem, three possible cases exist, depending upon the elastic limit of the material under consideration:—

- (1) The material may be of infinite strength;
- (2) Its elastic limit may be so high that the critical load, as determined by the theory of instability, is not sufficient to cause elastic break-down in the configuration of equilibrium;
- (3) Elastic break-down may occur, even in the position of equilibrium, at a load less than the critical value.

In the first case (which is, of course, purely ideal), the distortion due to loading will vanish when the loading is removed, and in this sense we may say that the

* In the problem of the tubular strut, the "favourite harmonic" is, of course, defined by that value of q which corresponds to a minimum value of \mathfrak{S} in equations (102) or (104).

material will never fail. The "curves of distortion," if we could determine their true shape, would probably be approximately of the form shown in fig. 9. The theoretical methods of this paper enable us to fix the position of A, the "point of bifurcation," but give no information as to the form of AB, beyond the fact that it cuts OA at right angles.* The other curves of the diagram will approach more and more closely the limiting form OAB as the initial value of the amplitude is decreased.

In the second case, we have the additional complication of elastic break-down under finite stress, which reduces the resistance of the material and causes the new "curves of distortion," shown by thick lines in fig. 10, to begin at certain points to fall away from the corresponding curves of fig. 9 (reproduced in fine lines for comparison); these points will lie on some line such as CD, cutting OA at a point *above* A, and it is clear that to the right of CD the curves of distortion refer to displacements which do not wholly vanish when the load is removed. Total collapse of the system will obviously occur at the points of maximum load on the curves of distortion, and the locus of these points, which is shown on the diagram by the dot-and-dash line EF, may be termed the "line of final collapse."

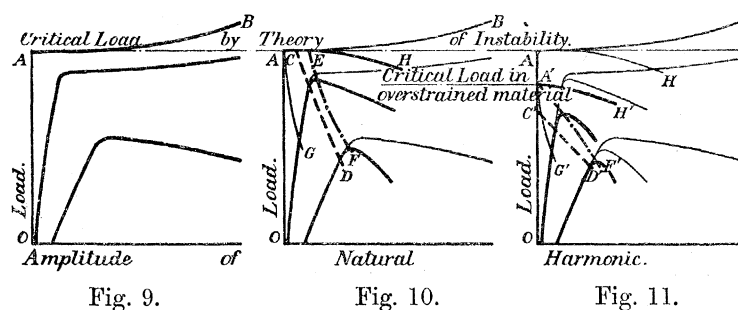


Fig. 9.

Fig. 10.

Fig. 11.

A knowledge of the true form of EF would enable us, when we are given the initial value of the amplitude, to predict the load at which the system will collapse; and these quantities could be connected by another curve AG, which would show at once whether the resistance of the system to collapse is seriously reduced by practical inaccuracies of form. A complete theory of any problem in elastic stability must yield information on this very important point, as well as an expression for the "critical load"; but in most cases more powerful methods would be needed for its derivation than are at present available. The investigation of the "critical load" is therefore not without utility, for although never realized in practice, this forms a limit which should be fairly closely approached when considerable accuracy is possible.

In our third case the "critical load," as deduced by theoretical methods, is more than sufficient to cause elastic break-down. We may proceed as before to draw hypothetical curves of distortion. The line C'D' (fig. 11), which corresponds to the

* It must not be assumed that AB is a horizontal straight line; in general, since the distorting effect of the applied stress-system, which varies as the deflection, increases less rapidly than the resistance, which varies as the curvature, AB will tend to rise from A.

line CD of fig. 10, will intersect OA at a point *below* A, and the other curves of distortion at correspondingly lower points. We have seen that the effect of local elastic breakdown upon fig. 10 was to deflect the curves of distortion from the forms which they would have assumed if the material had possessed indefinite strength; and it is clear that this deflection will begin at lower values of the loads in the present case. We may therefore expect curves of the type shown in thick lines in fig. 11, where the curves already obtained are reproduced in fine lines for comparison. As before, we may draw a line A'F' "of final collapse" through the points of maximum load on the curves of distortion, and connect the collapsing load with the initial value of the amplitude by another curve A'G'.

It is clear that the curves of distortion must tend to a limit which is no longer OAH, but some other curve OA'H', where OA', *the critical load under the new conditions*, is more than sufficient to produce elastic breakdown, but less than OA. We can see further that the curve A'G' is not likely to fall away from A' much more steeply than AG from A in fig. 10. The great weakness of short struts in practice, compared with EULER'S theoretical estimate, is now explained. Whereas long struts come within the conditions of fig. 10, the failure of short struts will be represented by fig. 11, and occurs at comparatively low stresses, not because practical imperfections have a greater effect upon the strength, but because OA', the true value of the critical load, is less than OA, the value which EULER'S theory would dictate.*

It is the rule, rather than the exception, that the critical load, as found by the ordinary theory of elastic stability, is more than sufficient in practice to produce elastic break-down. This may be readily seen in reference to any particular example. In the case of the tubular strut, fig. 10 is only applicable when the ratio of diameter to thickness is greater than 560 (for an average quality of mild steel), and for thicker tubes the critical load falls, apparently by a very considerable amount,† below the theoretical estimate. The determination of the critical load, in cases where this is more than sufficient to produce elastic break-down, is thus a problem of great importance, since it forms a limit which can never, under any circumstances, be exceeded. In the ordinary strut problem the determination can be effected without difficulty, and an apparently new field is thus indicated for research. The distinguishing feature of its problems is the dependence of the stress-strain relations upon the past history of the material, rendering absolutely necessary a method which follows the actual cycle of events up to the occurrence of collapse.

[* *Added May 11.*—Since this paper was written, the author's attention has been drawn to a dissertation by T. VON KÁRMÁN ('*Untersuchungen über Knickfestigkeit*,' Berlin, 1909), in which the forms of these "curves of distortion," for solid struts of practical dimensions, are deduced both from theory and from experiments. KÁRMÁN also gives a relation equivalent to that of equation (112).]

† Experiments conducted by the author upon seamless steel tubes showed failure under loads which were in every case little more than sufficient to produce "permanent set."

Stability of Short Struts.

This problem has been discussed elsewhere by the author,* and it will be noticed here only at sufficient length to indicate the directions in which further research is needed. We have to derive an expression for the collapsing load of a straight strut, when this is more than sufficient to cause elastic break-down of the material; and we proceed as before by considering three configurations of the strut: (1) before strain; (2) in a position of neutral equilibrium under uniform end-thrust; and (3) in a position of infinitesimal distortion from the second configuration.

For a first approximation we may say that cross-sections remain plane in the third configuration, so that the diagram of longitudinal compressive strain for any cross-section is as shown in the upper part of fig. 12; the horizontal line fg shows the

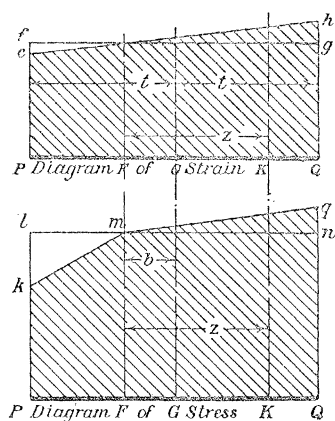


Fig. 12.

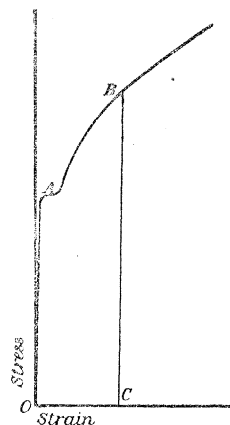


Fig. 13.

uniform strain of the second configuration. Then, if fig. 13 be the stress-strain diagram for a compression test of our material, and this uniform strain corresponds to a stress p which is represented by the point B, we see that to the right of the point F in fig. 12 the longitudinal compressive stress in the third configuration must be greater, and to the left less than p .

Now it is a well-known property of metals that if at any point B on the stress-strain diagram, beyond the elastic limit, we begin to decrease the load, the diagram is not retraced, but that we obtain a line BC which is parallel to OA.† It follows that the ratio

$$\frac{\text{decrease of stress}}{\text{decrease of strain}}$$

is still given by E, YOUNG'S Modulus for the material. On the other hand, the diagram shows that if we increase the load beyond B by an infinitesimal amount, the ratio

$$\frac{\text{increase of stress}}{\text{increase of strain}}$$

is a smaller quantity E' , which may be found from the slope of the diagram at B.

* 'Engineering,' August 23, 1912.

† A. MORLEY, 'Strength of Materials,' § 42.

We are considering an *infinitesimal* distortion in the third position, so that if we represent the increase of strain in fig. 12 by λz , the increase of longitudinal compressive stress to the right of F may be taken as $E'\lambda z$, and the decrease of this stress to the left of F as $E\lambda z$. Hence we obtain, for the section under consideration, the diagram of longitudinal stress which is shown in the lower part of fig. 12. The uniform stress of the second configuration is shown by the horizontal line ln , and it is a condition for neutral stability in the second configuration that no increase of thrust shall be required to maintain the distortion. If the cross-section of the strut is rectangular, of dimensions $a \times 2t$, it follows that the triangles lmk and qmn must be equal in area, or

$$\frac{FQ^2}{PF^2} = \frac{E}{E'} \dots \dots \dots (111)$$

This relation fixes the position of F on the cross-section, and in terms of the dimensions shown in fig. 12 we may write for the moment of resistance about G,

$$\begin{aligned} M &= \int_0^{b+t} E'\alpha\lambda z(z-b) dz + \int_{b-t}^0 E\alpha\lambda z(z-b) dz \\ &= \frac{2}{3}\alpha\lambda Et^3 + \frac{1}{3}\alpha\lambda(E-E')(b^3 - 3bt^2 - 2t^3). \end{aligned}$$

But if y is the deflection of the strut at the point G, in the infinitesimal distortion, we have, as in the ordinary theory of bending,

$$\lambda = -\frac{d^2y}{dx^2},$$

where x denotes the distance of the section from one end; and for equilibrium in the third configuration M must be equal to the bending moment due to thrust, or $2atpy$: hence,

$$py + \frac{d^2y}{dx^2} \times \frac{1}{3}Et^2 \left\{ 1 + \frac{1}{4} \left(1 - \frac{E'}{E} \right) \left(\frac{b^3}{t^3} - 3\frac{b}{t} - 2 \right) \right\} = 0 \dots \dots (112)$$

Equation (112) shows the modification which must be introduced, to take account of elastic break-down, into EULER'S equation

$$py + \frac{d^2y}{dx^2} \times \frac{1}{3}Et^2 = 0, \dots \dots \dots (113)$$

and it is easy to see that if l is the length, calculated from (113), of a strut which can just support the stress p , and l' the length as calculated from (112), then

$$\frac{l'^2}{l^2} = 1 + \frac{1}{4} \left(1 - \frac{E'}{E} \right) \left(\frac{b^3}{t^3} - 3\frac{b}{t} - 2 \right).$$

But, from (111),

$$\frac{t+b}{t-b} = \sqrt{\frac{E}{E'}}$$

so that, finally,

$$\frac{l'}{l} = \frac{2}{1 + \sqrt{\frac{E}{E'}}} \dots \dots \dots (114)$$

This result leads to a simple method by which the collapsing loads of short struts may be obtained graphically from the compressive stress-strain diagram. A full explanation is given in the paper to which reference has already been made, and a comparison, showing satisfactory agreement, is made with the results of experiments. One conclusion of some practical importance may be noticed: the curves of collapsing stress show that great ductility of material is by no means desirable in struts, the primary requisite being a high elastic limit.

Need for Further Research. Conclusion.

With slight modification, the theory just given for short struts might be applied to the problem of circular rings under radial pressure; but these appear to be the only cases in which we can at present discuss the stability of overstrained material. In any problem dealing with plates or shells distortion from the equilibrium position must introduce new stresses, in directions perpendicular to that of the stress which has caused elastic failure. The circular type of distortion in a tubular strut, for example, will introduce "hoop" stresses, and at present we have no knowledge of the corresponding stress-strain relations when "set" has occurred.

This and many other stability problems may be regarded as special cases of a general problem, viz., the determination of the changes of strain which occur when an infinitesimal stress-system, defined by principal stresses q, r, s , is impressed upon a material already overstrained by a simple stress p . The problem is not simple, and its solution would probably entail much theoretical and experimental work; but this would be justified by the importance, both for theory and practice, of its applications.

In conclusion, the author desires to express his indebtedness to Profs. LOVE and HOPKINSON, for valuable criticism and advice; to Mr. L. S. PALMER, for the photographs reproduced in fig. 2; and to Messrs. H. J. HOWARD and D. P. SCOTT, for assistance in the prosecution of the experiments described on p. 223. He also takes this opportunity of thanking Messrs. Stewarts and Lloyds, Ltd., of Glasgow, for gifts of very accurate steel tube for experimental purposes.

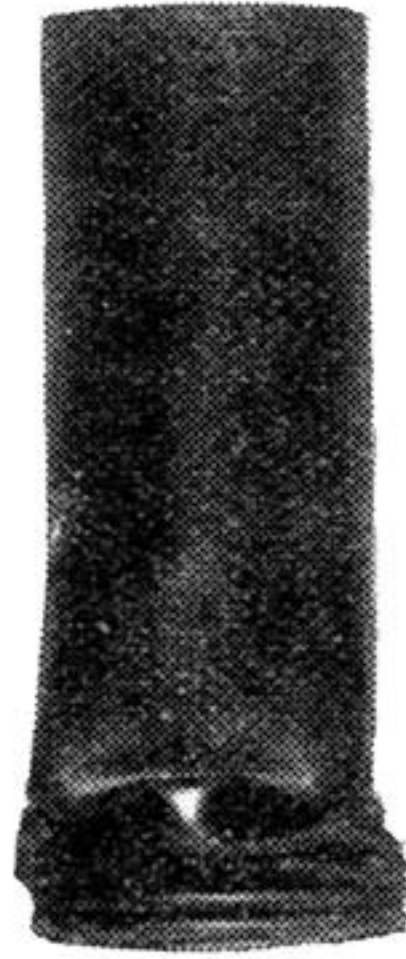


Fig. 2.